Introduction

Group theory is currently one of the most important mathematical disciplines, with manifold applications in

- mathematics (Galois theory, differential equations, geometry, etc.)
- physics (crystallography, solid state physics, high energy physics, gauge theories, phase transitions, general relativity, etc.)
- chemistry (molecular symmetries)

Question: what makes groups so ubiquitous?

Twofold origin of the group concept: notions of symmetry and (co)homology.

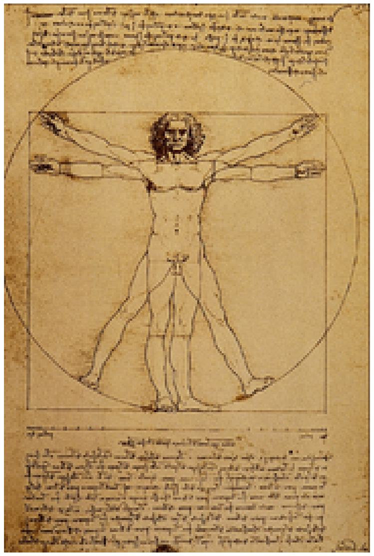
Homology is a topological notion characterizing the ”connectedness” of a manifold through a sequence of groups associated to it.

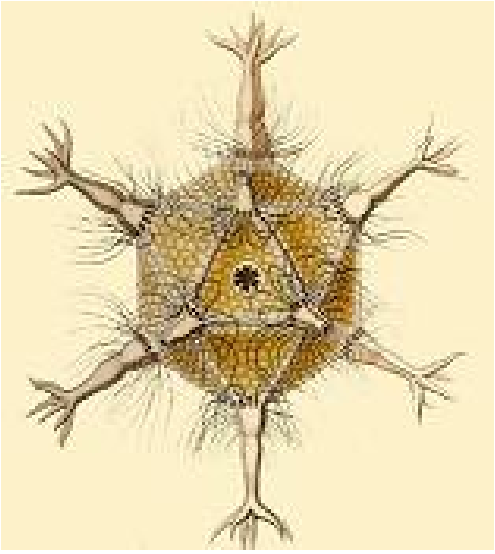
In physics, the notion of homology has important applications in the study of quantum systems, e.g. the Berry phase and topological quantum computing, gauge theories (instantons), general relativity (gravitational singularities) and string theory (branes).

The notion of homology can be extended to much more general situations (homological algebra).

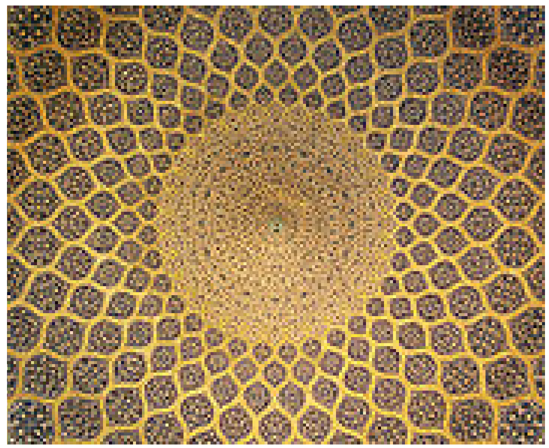
Symmetry: invariance under suitable transformations.

A classical example is the bilateral (mirror) symmetry of the human body, characteristic of a very large class of animals, the Bilateralia (to be contrasted with the five-fold rotational symmetry of echinoderms and the icosahedral symmetry of certain micro-organisms).





Similar symmetry patterns show up in artificial (man made) objects like buildings, furniture, decorations, etc.

Symmetry transformation can be composed, and groups describe the algebra of symmetries, hence group theory provides techniques to convert qualitative information into quantitative one.

Example: counting of phenomenological constants.

Small deformations of an elastic medium are described by Hooke's law

$$\sigma_{ij} = L_{ijkl} u_{kl}$$

with u_{ij} and σ_{ij} denoting the deformation and the stress tensor.

$3^4 = 81$ different L_{ijkl} components, related by Onsager's reciprocity law

$$L_{ijkl} = L_{jikl} = L_{klij}$$

Question: how many independent coefficients L_{ijkl} (elastic moduli) characterize a given medium?

Answer: 2 for isotropic media, 21 for a triclinic crystal.

Explanation: crystalline structure characterized by its symmetries, part of which form a group of matrices, the so-called point group.

For a crystal with point group G, there are

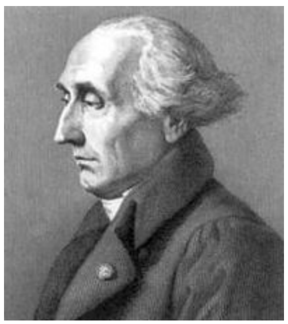
$$\frac{1}{8}\sum_{g\in G}\left\{\mathrm{Tr}(g)^4 + 2\mathrm{Tr}(g)^2\,\mathrm{Tr}\big(g^2\big) + 3\mathrm{Tr}\big(g^2\big)^2 + 2\mathrm{Tr}\big(g^4\big)\right\}$$

independent elastic moduli (in any dimension).

In complete generality, the scalar free energy is an invariant polynomial of the (symmetric) deformation tensor, hence the answer equals the number of independent invariants of the symmetrized square of the point group (the above formula gives the number of quadratic invariants, appropriate in the linear case described by Hooke's law).

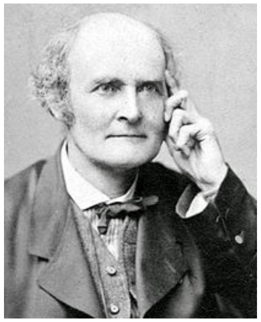
1 Historical highlights

Antiquity: application of symmetry principles in geometry (Euclid of Alexandria, Archimedes of Syracuse, etc.), classification of platonic solids.



J.-L. Lagrange (1771): solubility of polynomial equations.

É. Galois (1832): Galois theory.



A. Cayley (1854): abstract group concept.

É. Mathieu (1861,1873): discovery of the Mathieu groups.

F. Klein (1873): Erlangen program (classification of geometries via symmetry principles).

S. Lie (1871-1893): continuous transformation groups.

H. Poincaré (1882): homology groups, uniformization.

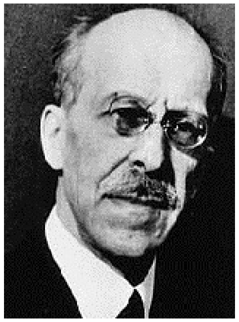
É. Picard (1883): differential Galois theory.

D. Hilbert (1888): theory if invariants, homological algebra.

  W. Killing (1880-1890) and É. Cartan (1894): classification of simple Lie-groups and their Lie-algebras.

 G.F. Frobenius (1896): representations, group characters.

 W. Burnside (1903): finite groups.



I. Schur (1904): projective representations.

A. Haar (1933): invariant integrals.

1966-1976: sporadic simple groups.

Group theory in physics

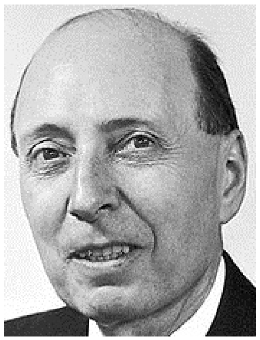
Renaissance: symmetry principles in statics ("Epitaph of Stevinus").

L. Euler (1765): movement of rigid bodies.

E.S. Fedorov (1891), L. Schönflies (1891) and W. Barlow (1894): classification of crystal structures in 3D.

H. Poincaré (1900): symmetries of Maxwell's equations.

E. Noether (1915): symmetries vs conservation laws.



E. Wigner: symmetries in quantum physics (1933), classification of relativistic wave equations (1947).

C.N. Yang és R. Mills (1954): local gauge symmetries.

M. Gell-Mann (1963): ”eightfold way” (basis of the quark model).

D. Shechtman (1982): discovery of quasi-crystals.

2 Fundamental concepts

Question: What is a group? How to compare groups?

A group is a set of elements with a suitable binary operation.

Binary operation: rule that assigns to two elements of a set a well-defined third element of that same set.

Examples: addition and multiplication of (integer, rational, real, complex, hypercomplex, p-adic, etc.) numbers, greatest common divisor and lowest common multiple of integers, addition and cross product of vectors (but not their scalar product), addition and multiplication of linear operators /polynomials/matrices.

Multiplicative infix notation: for a binary operation on the set X, we denote by $x \star y$ (or simply by xy) the element obtained by applying the operation to the elements $x, y \in X$.

A binary operation (on the set X) is

associative, if for any $x, y, z \in X$

$$x(yz) = (xy)z$$

commutative, if for any $x, y \in X$

$$xy = yx$$

unital, if there exists $\mathbf{1}_X \in X$ (the identity) such that for all $x \in X$

$$\mathbf{1}_X x = x \mathbf{1}_X = x$$

A **group** is a set G of elements together with an associative and unital binary operation (the 'product'), such that for all $x \in G$ there exists $x^{-1} \in G$ (the inverse of x) for which $xx^{-1} = x^{-1}x = \mathbf{1}_G$.

The order of a group is the cardinality of its set of elements. A group is a finite if its order is finite, i.e. a positive integer.

A group is said to be Abelian if its product is commutative.

Abelian groups have very special properties (they form a subvariety of the algebraic variety of groups).

The groups G and H are isomorphic, denoted $G \cong H$, if there exists a bijective (i.e. one-to-one) map $\phi : G \to H$ that preserves products, i.e.

$$\phi(xy) = \phi(x)\,\phi(y)$$

for all $x, y \in G$ (such a map is called an isomorphism).

The relation of being isomorphic is reflexive ($G \cong G$), symmetric ($G \cong H$ implies $H \cong G$) and transitive ($G \cong H$ and $H \cong K$ implies $G \cong K$).

Isomorphy principle: isomorphic groups cannot be distinguished from each other by algebraic means (they have the same algebraic structure).

Group theoretic properties are the same for isomorphic groups, e.g. the orders of isomorphic groups are the same: $G \cong H$ implies $|G| = |H|$.

An automorphism of a group is an isomorphism of the group with itself, and the collection $\mathrm{Aut}(G)$ of all automorphism of a group G is itself a group, the automorphism group ('symmetry group') of G, with product the composition of maps.

A collection H of elements of a group G is called a subgroup, denoted $H < G$, if it contains the identity element of G and the product of any two of its elements, as well as the inverse of all its elements.

The relation of being a subgroup is an ordering. In particular, every subgroup is a group, and a subgroup of a subgroup is itself a subgroup: if $K < H$ and $H < G$, then $K < G$.

Generalizations of the group concept:

- relaxing the existence of inverses leads to monoids (with applications to automata theory & linguistics, renormalization, etc.);
- relaxing the associativity of the product results in quasi-groups (combinatorial applications like latin squares, aka. sudoku);
- a partially defined product gives groupoids (with applications to topology, the description of quasi-crystals, etc.);
- theoretical physics $\rightsquigarrow$ quantum groups, supergroups, ...

Recommended reading:

A.A. Kirillov : Elements of the theory of representations, Springer (1976).

J.L. Alperin and B. Bell : Groups and representations, Springer (1995).

D. Robinson : A course in the theory of groups, Springer (1993).

W. Magnus, A. Karrass and D. Solitar: Combinatorial group theory, Interscience Publishers (1966).

3 Examples of groups

3.1 Additive and unit groups of rings

Consider a (unital) ring R , i.e. a set with two associative and unital binary operations - addition ('+') and multiplication ('$\cdot$') - that satisfy

1. addition is commutative, and each $a \in R$ has an additive inverse, denoted $-a$, such that $a+(-a)=(-a)+a=0$, where 0 denotes the additive identity (the zero element of R);

2. the distributive laws $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ hold for any $a, b, c \in R$.

Examples:

1. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ of integer, rational, real and complex numbers, with the usual addition and multiplication, are all commutative rings. But the set $2\mathbb{Z}+1$ of odd integers is not a ring (since the sum of two odd numbers is even), nor is the set $\mathbb{N}$ of natural numbers (since addition is not unital).

   Remark. The notion of ring has been abstracted from the notion of number systems, which forms the basis of mathematical thought. As we have all learned early on, one can add and multiply together numbers to get numbers again, and these operations are associative and commutative. That natural numbers - which count the number of elements in a collection, hence are the most immediate for practical use - are insufficient has been known since antiquity (even the need for irrational numbers has been known to the Greeks), and this led to the notions of negative, rational and real numbers, while the need for complex numbers in the treatment of many problems became evident during the 18th century.

The first to study number rings was R. Dedekind in 1872. He was followed by D. Hilbert 20 years later, while the definitive axiomatic formulation was given by E. Noether in 1921, who began to develop a general theory of rings together with Krull, van der Waerden, Macaulay, Cohen, ...

2. For any positive integer n, the so-called modulus, the set

$$\mathbb{Z}/n\mathbb{Z} = \{n\mathbb{Z} + k \mid 0 \leq k < n\}$$

whose elements consist of collection of integers whose remainder upon integer division by n are equal, is a commutative ring, the ring of residue classes modulo n, with sum and product the residue class containing the pairwise sums (resp. products) of the summands (resp. factors).

Remark. The ring $\mathbb{Z}/n\mathbb{Z}$ can be described alternatively as the set $\{0, 1, \ldots, n-1\}$, with addition and multiplication modulo n, i.e. the sum/product of two elements is the remainder mod n of their sum/product.

3. For a commutative ring R and a set X, the set $R(X)$ of R-valued functions on X, i.e. maps from X to R, is a commutative ring with the pointwise operations

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)\,g(x)$$

for $x \in X$. If both R and X have some kind of geometric structure (e.g. topological, differentiable, complex analytic, etc.), then the structure preserving maps (e.g. continuous, differentiable, holomorphic, etc.) form themselves a ring, and the algebraic structure of this ring reflects the geometric structure $\rightsquigarrow$ algebraic geometry.

4. For a commutative ring R and a positive integer n, a polynomial in n indeterminates is a map from $\mathbb{Z}_+^n = \{(\alpha_1, \ldots, \alpha_n) \,|\, \alpha_1, \ldots, \alpha_n \in \mathbb{Z}_+\}$ (the set of n-tuples of non-negative integers) into R, with finite support, i.e. which takes on non-zero values only at finitely many arguments. The addition of polynomials is defined pointwise, while the product of the polynomials P and Q reads

$$(PQ)(\alpha_1, \ldots, \alpha_n) = \sum_{0 \leq \beta_i \leq \alpha_i} P(\beta_1, \ldots, \beta_n)\, Q(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$$

With these operations, polynomials form the ring $R[x_1, \ldots, x_n]$. The indeterminates, defined by $x_i(\alpha_1, \ldots, \alpha_n) = \delta_{\alpha_i,1} \prod_{j \neq i} \delta_{\alpha_j,0}$ for $i = 1, \ldots, n$, are special polynomials that generate the whole ring.

5. Informally, a matrix A is a rectangular array of numbers, with the matrix element A_{ij} denoting the number found at the intersection of the i^{th} row and j^{th} column. One can add matrices A and B of the same shape (i.e. with the same number of rows and columns) element-wise, that is $(A+B)_{ij} = A_{ij} + B_{ij}$, while their product AB, only defined in case the number m of columns of A equals the number of rows of B, reads $(AB)_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$. More generally, the elements of a matrix may come from any ring. A square matrix of size n is one with n rows and columns.

   The set of square matrices of size n over the commutative ring R is itself a ring $\mathrm{Mat}_n(R)$, which is not commutative unless $n=1$.

The additive group of a ring R is the set of elements of R with the operation of addition (always an Abelian group).

A unit of a ring R is an element $a \in R$ having a multiplicative inverse, i.e. an element $b \in R$ for which

$$ab = ba = \mathbf{1}_R$$

where $\mathbf{1}_R$ denotes the multiplicative identity ('one') of R: a multiplicative inverse is unique if it exists, and is usually denoted a^{-1}.

The units of a ring R form a group with the operation of multiplication, the unit group $R^\times$ of R (a ring whose unit group contains all of its nonzero elements is called a field, e.g. $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$).

Examples of unit groups:

1. $\mathbb{Z}^\times = \{\pm 1\}$;

2. $(\mathbb{Z}/n\mathbb{Z})^\times = \{n\mathbb{Z}+k \mid 0 \leq k < n \text{ coprime to } n\}$, the group of prime residues mod n;

3. $R(X)^\times = R^\times(X)$;

4. $R[x_1, \ldots, x_n]^\times = R^\times$, the set of constant polynomials with values in $R^\times$, that consists of those polynomials $P \in R[x_1, \ldots, x_n]$ for which $P(0, \ldots, 0) \in R^\times$ and $P(\alpha_1, \ldots, \alpha_n) = 0$ if any $\alpha_i > 0$.

5. $\mathrm{Mat}_n(R)^\times = \mathsf{GL_n}(R) = \{A \in \mathrm{Mat}_n(R) \mid \det A \in R^\times\}$, the general linear group over R, consists of all invertible n-by-n matrices.

3.2 Matrix groups

Besides the general linear group $\mathsf{GL_n}(R)$, several other subsets of $\mathrm{Mat}_n(R)$ form a group with the operation of matrix multiplication:

1. the set

$$\Delta_n(R) = \{A \in \mathsf{GL_n}(R) \mid A_{ij} = 0 \text{ if } i \neq j\}$$

   of diagonal matrices is an Abelian group;

2. those matrices in which each row and column contains exactly one nonzero entry, the monomial matrices, form a group $\mathsf{M}_n(R)$;

3. the set

$$\Pi_n = \{A \in \mathsf{M}_n(R) \mid A_{ij} \neq 0 \text{ implies } A_{ij} = \mathbf{1}_R\}$$

of permutation matrices is also a group (note that every monomial matrix is the product of a diagonal and a permutation matrix);

4. the set

$$\mathsf{O}_n(R) = \left\{A \in \mathsf{GL_n}(R) \mid A^{-1} = A^{\mathsf{T}}\right\}$$

of orthogonal matrices (where A^{T} denotes the *transpose* of a matrix $A \in \mathrm{Mat}_n(R)$, with entries $(A^{\mathsf{T}})_{ij} = A_{ji}$), and more generally

$$\mathsf{O}_{p,q}(R) = \left\{A \in \mathsf{GL_{p+q}}(R) \mid A^{-1} = \eta_{p,q}^{-1} A^{\mathsf{T}} \eta_{p,q}\right\}$$

with $\eta_{p,q}$ denoting the diagonal matrix having the first p diagonal entries equal to $\mathbf{1}_R$, and the remaining q entries equal to $-\mathbf{1}_R$;

5. the set

$$\mathsf{Sp}_{2n}(R) = \left\{ A \in \mathsf{GL}_{2\mathsf{n}}(R) \mid A^{-1} = \mathsf{J}_n^{-1} A^{\mathsf{T}} \mathsf{J}_n \right\}$$

of symplectic matrices, where J_n is the block-diagonal matrix made up of n copies of the *Pauli-matrix*

$$\mathtt{i}\sigma_2 = \begin{pmatrix} 0 & \mathbf{1}_R \\ -\mathbf{1}_R & 0 \end{pmatrix}$$

6. those subsets of all the above whose elements satisfy $\det A = \mathbf{1}_R$, e.g. the special linear and orthogonal groups

$$\mathsf{SL}_n(R) = \{ A \in \mathsf{GL}_\mathsf{n}(R) \mid \det A = \mathbf{1}_R \}$$

$$\mathsf{SO}_n(R) = \{ A \in \mathsf{O}_n(R) \mid \det A = \mathbf{1}_R \}$$

3.3 Symmetric and alternating groups

A permutation is a bijective self-map of a set onto itself ('reshuffling').

The product of permutations is the composition of the corresponding maps: this is an associative and unital binary operation, with the trivial permutation $\mathbf{id}_X : x \mapsto x$ as identity element.

Inverse of a permutation: inverse map (once again bijective).

The collection of all permutations of a (finite) set X forms a group $\mathsf{Sym}(X)$, the symmetric (*NOT symmetry*!) group over X.

$\mathsf{Sym}(X)$ is not commutative in case $|X| > 2$, and has order

$$\left|\mathsf{Sym}(X)\right| = |X|!$$

Transposition: interchange of two elements.

Any permutation can be decomposed (in many ways) into a product of transpositions: while their number may change, whether there is an even or odd number of them characterizes the permutation, allowing to distinguish odd and even permutations.

The product of even permutations is even, hence even permutations form themselves a group $\mathsf{Alt}(X)$, the alternating group over X.

The symmetric groups $\mathsf{Sym}(X)$ and $\mathsf{Sym}(Y)$ are isomorphic precisely when $|X| = |Y|$, hence it is enough to consider the symmetric groups $\mathbb{S}_n = \mathsf{Sym}(\{1, \ldots, n\})$ (resp. alternating groups $\mathbb{A}_n = \mathsf{Alt}(\{1, \ldots, n\})$) of degree n, isomorphic to Π_n and $\mathsf{S}\Pi_n$ respectively.

A cycle (orbit) of a permutation is a set of points that are taken into each other by successive applications of the permutation (a fixed point is a cycle of length 1).

The cycles of a permutation $\pi \in \mathsf{Sym}(X)$ partition the set X, i.e. any two are either equal or disjoint, and each point of X belongs to some cycle.

A permutation is called cyclic if it has only one cycle of length greater than one, and the length of this cycle is its order (or period).

Every permutation can be decomposed into a product of cyclic ones.

A cyclic permutation $\pi \in \mathsf{Sym}(X)$ is uniquely determined by the sequence

$$x_1, x_2 = \pi(x_1), \ldots, x_n = \pi(x_{n-1}), \ldots$$

where x_1 is any element of the non-trivial cycle of π (cycle notation).

3.4 Geometric symmetry groups

Rigid motion: mapping of Euclidean space onto itself that preserves the distance of points (Euclidean isometry).

Types of rigid motions: translations, rotations, reflections, and different composites of the above.

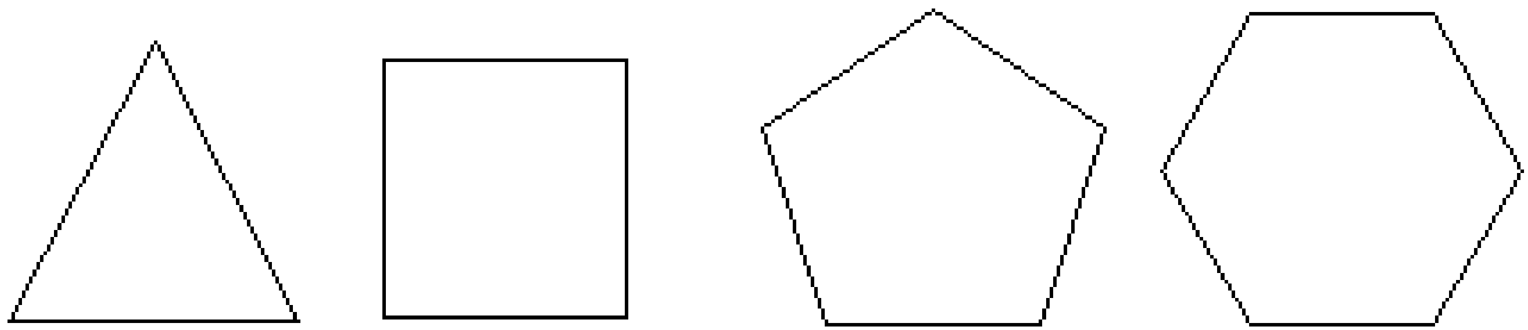
Symmetry of a geometric figure: rigid motion mapping the figure (as a set of points) onto itself.

For example, any rotation around an axis passing through the center of a sphere is a symmetry of the sphere (but the sphere has reflection symmetries as well).

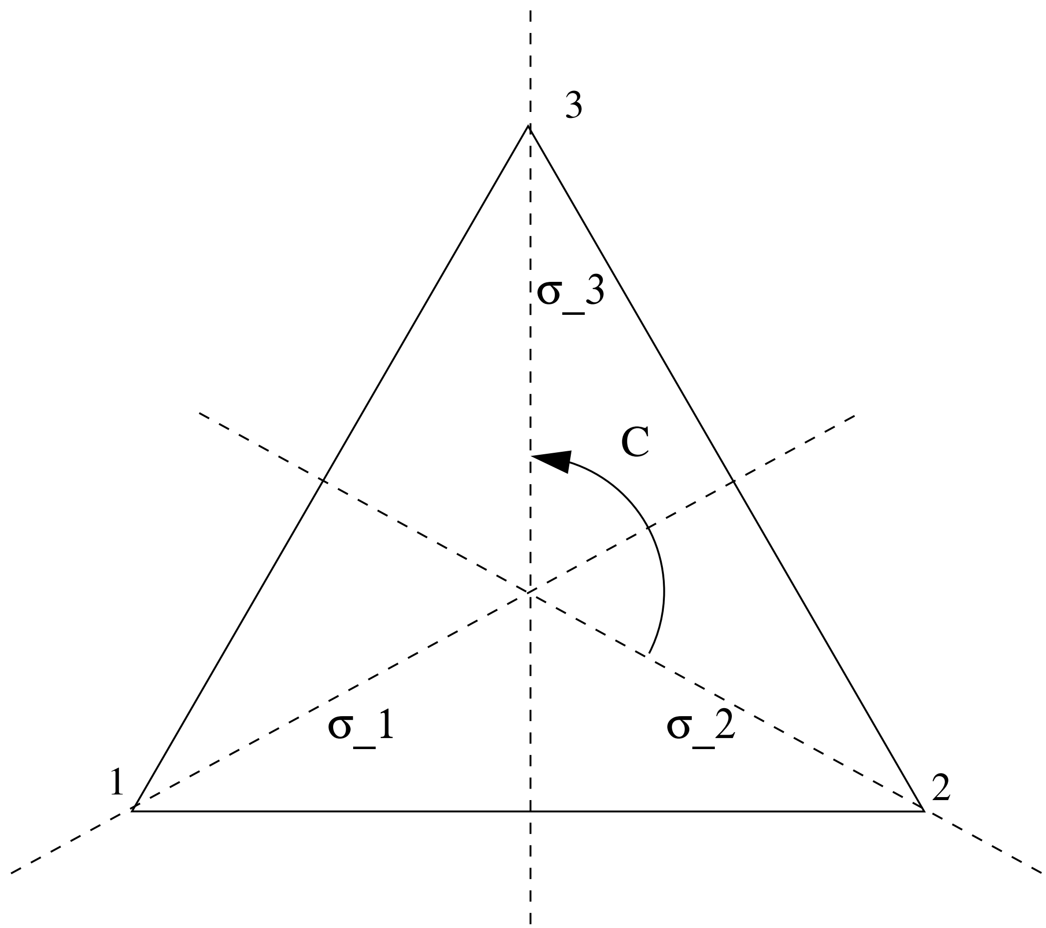
Regular polygon: convex plane figure all of whose sides are congruent (i.e. have equal length), and angles between neighboring sides are equal.

Medians (edge bisectors) of a regular polygon all meet in a single point, the center of the polygon.

For each integer $n > 2$ there is exactly one regular n-gon (up to similarity).



The symmetries of a regular n-gon, composed of rotations around the center (by multiples of $2\pi/n$) and reflections across lines passing through the center and some vertex, form the dihedral group $\mathbb{D}_n$ of degree n.



Since there are n different reflection axes (the medians for odd n, and the medians together with the diagonals passing through opposite vertices for even n) and n different rotations, the order of the dihedral group is $|\mathbb{D}_n| = 2n$.

For finite groups, the group structure can be neatly described using the Cayley table.

	$\mathbf{1}_G$	$\dots$	h	$\dots$
$\mathbf{1}_G$	$\mathbf{1}_G$	$\dots$	h	$\dots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
g	g	$\dots$	gh	$\dots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

Since symmetries map the polygon onto itself, any set of distinguished subfigures (like vertices, edges, medians, etc.) is also mapped onto itself by a symmetry transformation.

As a consequence, each symmetry transformation induces a permutation of any choosen set of distinguished subfigures (be it vertices, edges, medians, etc.), which can prove handy for the computation of the Cayley table.

Remark: since the set V of vertices of a regular n-gon has cardinality n, the above correspondence is bijective only for $n=3$, when

$$\left|\mathsf{Sym}(V)\right| = n! = 2n = \left|\mathbb{D}_n\right|$$

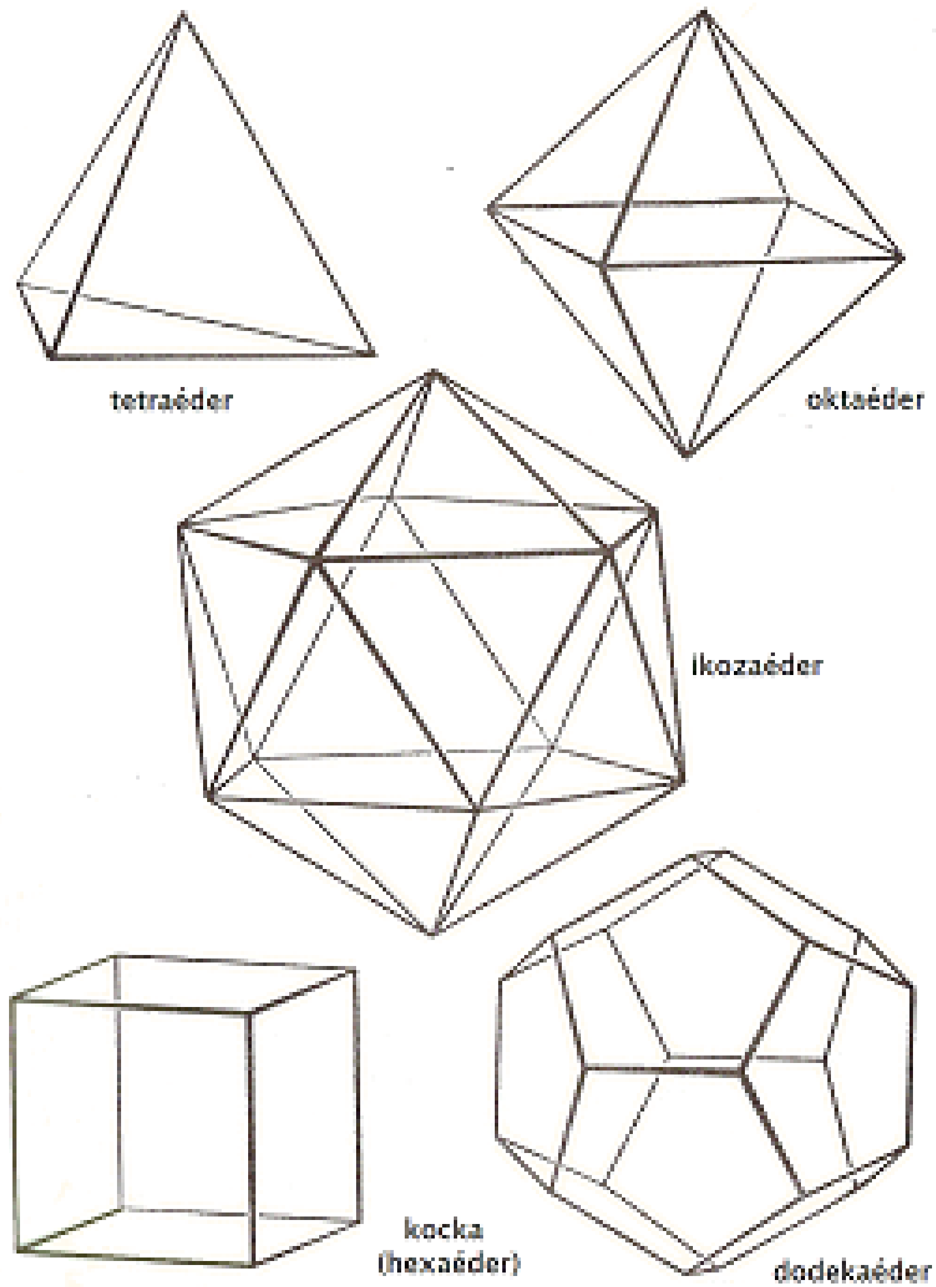
	1	C	C^2	σ_1	σ_2	σ_3	
1	1	C	C^2	σ_1	σ_2	σ_3	$()$
C	C	C^2	1	σ_3	σ_1	σ_2	$(1,2,3)$
C^2	C^2	1	C	σ_2	σ_3	σ_1	$(1,3,2)$
σ_1	σ_1	σ_2	σ_3	1	C	C^2	$(2,3)$
σ_2	σ_2	σ_3	σ_1	C^2	1	C	$(1,3)$
σ_3	σ_3	σ_1	σ_2	C	C^2	1	$(1,2)$

Multiplication table of $\mathbb{D}_3$ and the induced permutation of vertices.

A convex spatial figure all of whose bounding facets are congruent regular polygons is called a Platonic solid (regular polyhedron).

There are five different Platonic solids (up to similarity):

1. tetrahedron (4 triangles);
2. octahedron (8 triangles);
3. icosahedron (20 triangles);
4. cube (6 squares);
5. dodecahedron (12 pentagons).


tetraéder
oktaéder
ikozaéder
kocka
(hexaéder)
dodekaéder

The symmetry groups of regular polyhedra are:

- tetrahedral group $\mathbf{T} \cong \mathbb{A}_4$ (tetrahedron),
- octahedral group $\mathbf{O} \cong \mathbb{S}_4$ (octahedron $\rightleftharpoons$ cube)
- icosahedral group $\mathbf{I} \cong \mathbb{A}_5$ (icosahedron $\rightleftharpoons$ dodecahedron).

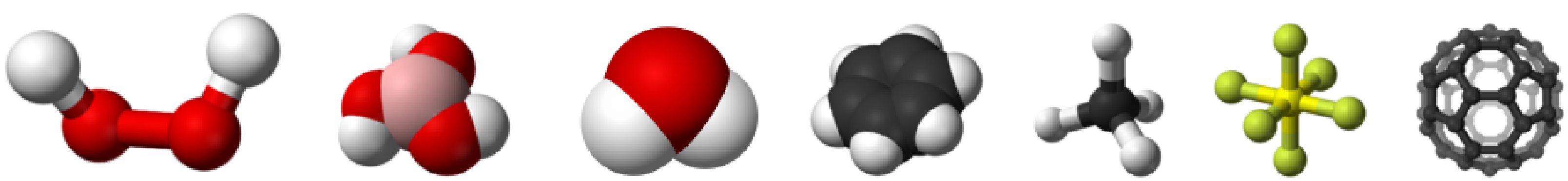
Regular convex polytopes in 4D: simplex (5-cell), orthoplex (16-cell), hypercube (8-cell), 600-cell, 120-cell, 24-cell (with 1152 symmetries).

Only 3 regular polytopes in dimensions >4: the simplex , the orthoplex (cross-polytope) and the hypercube.

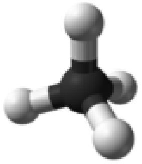
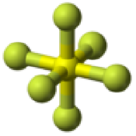
3.5 Molecular symmetry groups

Charge density in molecules invariant under a finite group (point group) of geometric transformations, leading to restrictions on

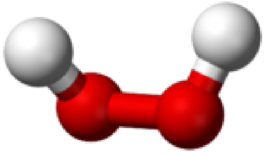
1. the structure of the molecular spectrum (Wigner, Tisza);
2. electromagnetic characteristics (dipole and magnetic moments);
3. chemical properties.

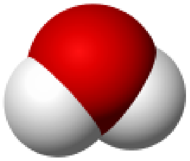


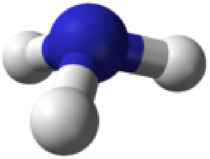
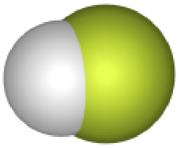
Point groups in 3D: 7 polyhedral groups

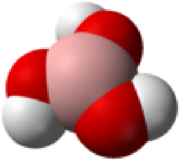
1. $\mathbf{T} \cong \mathbb{A}_4$ (chiral tetrahedral)
2. $\mathbf{T}_\mathrm{d} \cong \mathbb{S}_4$ (tetrahedral), e.g. methane 
3. $\mathbf{T}_\mathrm{h} \cong \mathbb{A}_4 \times \mathbb{Z}_2$ (pyritohedral)
4. $\mathbf{O} \cong \mathbb{S}_4$ (chiral octahedral)
5. $\mathbf{O}_\mathrm{h} \cong \mathbb{S}_4 \times \mathbb{Z}_2$ (octahedral), e.g. sulfur hexafluoride SF_6 
6. $\mathbf{I} \cong \mathbb{A}_5$ (chiral icosahedral)
7. $\mathbf{I}_\mathrm{h} \cong \mathbb{A}_5 \times \mathbb{Z}_2$ (icosahedral), e.g. C_{60} fullerene 

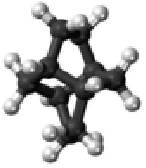
+ 7 infinite families of axial groups

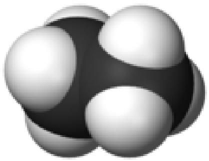
1. $C_n \cong \mathbb{Z}_n$ (cyclic), e.g. hydrogen peroxide  $(n=2)$

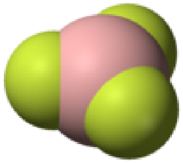
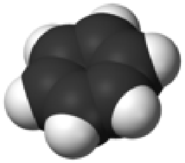
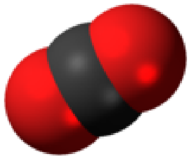
2. $C_{nv} \cong \mathbb{D}_n$ (pyramidal), e.g. water  $(n=2)$,

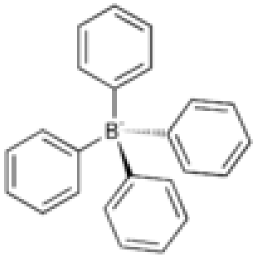
   ammonia  $(n=3)$, hydrogen fluoride  $(n=\infty)$

3. $C_{nh} \cong \mathbb{Z}_n \times \mathbb{Z}_2$, e.g. boric acid  $(n=3)$

4. $D_n \cong \mathbb{D}_n$ (dihedral), e.g. twistane $C_{10}H_{16}$  $(n=2)$

5. $D_{n\mathrm{d}} \cong \mathbb{D}_{2n}$ (anti-prismatic), e.g. ethane $\mathtt{C_2H_6}$  $(n=3)$

6. $D_{n\mathrm{h}} \cong \mathbb{D}_n \times \mathbb{Z}_2$ (prismatic), e.g. boron trifluoride  $(n=3)$, benzene  $(n=6)$, carbon dioxide  $(n=\infty)$

7. $S_{2n} \cong \mathbb{Z}_{2n}$, e.g. tetraphenylborate  $(n=4)$

3.6 Crystalline symmetry groups

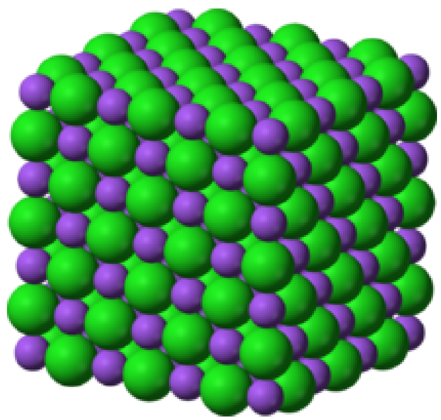
Some homogeneous substances exhibit anisotropic (direction dependent) behavior on a macroscopic scale: e.g. mechanical, optical, electric, etc. properties of crystals.

At the phenomenological level, the material characteristics (like permittivity, heat conductivity, elastic moduli, etc.) are tensorial quantities (rather than scalars).

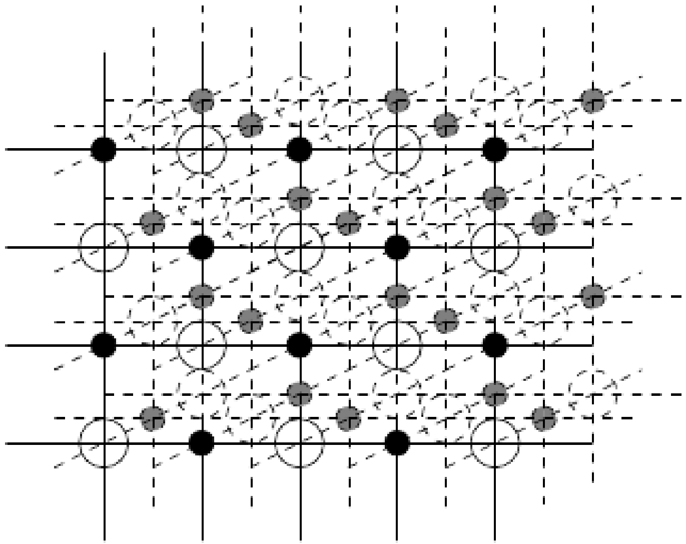
Since both microscopic homogeneity/isotropy and unordered microscopic inhomogeneity/anisotropy leads to isotropic behavior, macroscopic anisotropy is a consequence of ordered microscopic inhomogeneity/anisotropy.

Ordered microscopic anisotropy is realized in ferro- and ferrimagnetic materials, while ordered inhomogeinity in (quasi-)crystals.

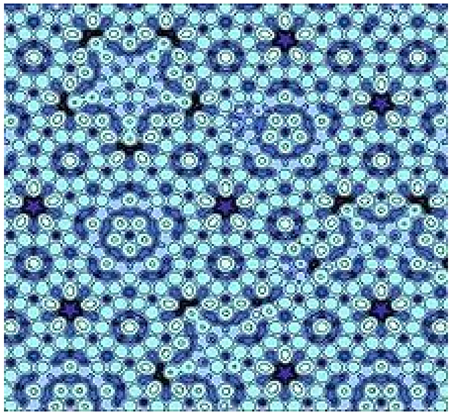
Macroscopic order can arise from discrete translational symmetries: a periodic structure in space exhibits ordered inhomogeinity.



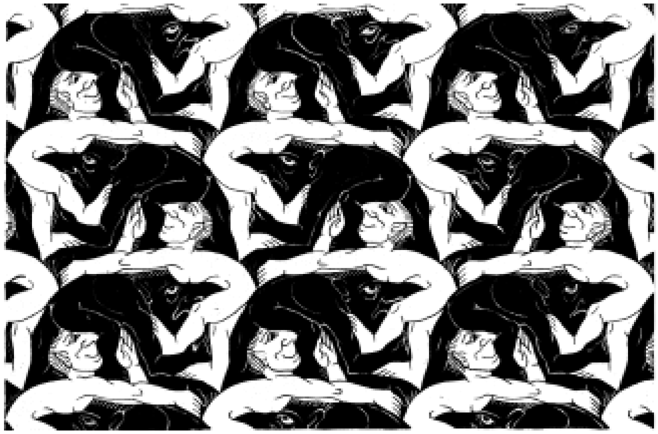
Crystalline substance: microscopic components (atoms/molecules/ions) distributed periodically in space, localized (in the absence of defects) around the lattice points of a 3D periodic lattice (the crystal lattice).



Quasi-crystal: aperiodic structure exhibiting long range order.



Space group: full symmetry group of the crystal, taking into account the symmetries of its microscopic constituents (composed of translations, rotations, reflections, inversions and various combinations of the above).



Translation subgroup: group of translations taking the crystal into itself.

Point group: finite group describing the rotational and reflection symmetries of the crystal.

Crystal structures are grouped into crystal classes, families and systems according to their point groups and translation subgroups.

Order of a transformation: smallest positive integer N such that the $N^{\mathtt{th}}$ power of the transformation is the identity.

Crystallographic restriction: the number of integers coprime to the order of any element of the point group cannot exceed the dimension of space (valid for periodic structures, not quasi-crystals).

dimension	$2,3$	$4,5$	$6,7$
allowed values	$\{2,3,4,6\}$	$\cup\{5,8,10,12\}$	$\cup\{7,9,14,18\}$

Consequence: there are only finitely many different crystal structures in any space dimension.

Classification results

dimension	2	$3^{\dagger}$	$4^{\ddagger}$	5^{*}	6^{*}
# crystal systems	4	7	33	59	251
# point groups	10	32	227	955	7104
# space groups	17	230	4894	222097	28934974

†: Fedorov (1891), Schönflies (1891) and Barlow (1894).

‡: Brown, Bülow and Neubüser (1978).

$*$: Plesken and Schulz (2000).

3.7 Space-time symmetries

Kinematics: description of the movement of material bodies, i.e. of the time evolution of their mutual emplacements (relative distances).

Frame of reference: system of bodies with known relative motions.

The relative motions of all bodies of the Universe are completely determined by their motions with respect to a specific frame of reference.

Question: are there reference frames that are more useful than others?

Answer: inertial frames, in which the inertial motion of isolated (not interacting with the rest of the Universe) bodies is uniform translation.

Remark: inertial frames move at constant speed relative to each other.

In classical (Newtonian) mechanics

- physical space is 3D Euclidean
- time is a 1D continuum
- forces act instantly, without any delay (action at a distance).

Galileo's relativity principle: not only the law of inertial motion, but all mechanical laws look the same in every inertial frame.

It follows that in inertial frames,

- both space and time are homogeneous, without a preferred origin;
- space is isotropic, i.e. there are no preferred directions;
- any reference frame obtained via a boost (constant speed uniform translation) from an inertial one is itself inertial.

These are universal symmetries governing the structure of natural laws: the laws of classical mechanics are the same anywhere and at anytime, irrespective of the orientation and of the inertial frame chosen.

The symmetries of classical mechanics form the Galilei group $\mathcal{G}$, whose elements consist of

- space translations (3 parameters)
- time translations (1 parameter)
- spatial rotations (3 parameters)
- (Galilean) boosts (3 parameters)
- discrete reflection symmetries

Noether's theorem: to each one-parameter group of continuous symmetries of a physical system corresponds a conserved quantity.

Galilean symmetries correspond to universal first integrals.

space translations	(linear) momentum
spatial rotations	angular momentum
time translations	energy
(Galilean) boosts	center of mass

The existence of universal symmetries is corroborated by the classical conservation laws (energy, momentum, etc.)!

Einstein: Galileo's relativity principle holds for all laws of physics (including those of electrodynamics), not only those of classical mechanics, but there is a limit speed c for the propagation of physical causes, i.e. there is no action at a distance.

Remark: the limit speed c equals the speed of light in vacuum.

Poincaré: the symmetry group of Maxwell's equations is the isometry group of 4D (homogeneous and isotropic) Minkowski space, the so-called Poincaré group $\mathcal{P}$, and not the Galilei group.

Remark: the Galilei group may be obtained from the Poincaré group by a limiting procedure (Wigner-Inönü contraction) when $c \to \infty$.

The true symmetry group of physical laws (at arbitrary speeds, but neglecting the effects of gravity) is the Poincaré group $\mathcal{P}$, containing

- space-time translations (4 parameters)
- space-time rotations, including the spatial rotations and the Lorentz-boosts (6 parameters)

Two relativistic first integrals: a 4D vector (four-momentum) and a 4D antisymmetric tensor (angular momentum).

In general relativity, flat Minkowski space-time is replaced by a curved manifold (a solution of Einstein's equations), and the Poincaré group by its isometry group (e.g. the de Sitter group).