Lie groups

1 Lie groups and their parametrization

A topological group is a group G that is at the same time a topological space such that for $g \in G$ both the left translations

$$\begin{aligned} \boldsymbol{\lambda}_g : G &\to G \\ h &\mapsto gh \end{aligned}$$

and the inversion map

$$\begin{aligned} \boldsymbol{\iota}_G : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are continuous.

A manifold is a topological space that is locally Euclidean: it is covered by open sets W, each homeomorphic to an open subset $\mathcal{U} \subseteq \mathbb{R}^n$, where $\mathbb{R}^n$ denotes n-dimensional Euclidean space, i.e. the set of n-tuples of real numbers with the usual topology associated to the Euclidean metric.

Remark. A local homeomorphism $\boldsymbol{\alpha}_W : W \to \mathcal{U}$ is called a local chart, since it allows to parametrize each point $x \in W$ by real-valued curvilinear coordinates, the components of $\boldsymbol{\alpha}_W(x)$.

Overlapping local charts give rise to different local parametrizations of one and the same point, related to each other by transition functions.

If the positive integer n is the same for all local charts, then n is called the dimension of the space.

An n-parameter Lie group is a topological group that is locally Euclidean of dimension n. The local charts allow to parametrize (locally) the group elements: to a group element $g \in W$ is associated its parameter vector

$$\vec{\boldsymbol{\alpha}}(g) = (\boldsymbol{\alpha}_1(g), \ldots, \boldsymbol{\alpha}_n(g))$$

For example, 3D space translations $\tau: \mathbb{R}^3 \to \mathbb{R}^3$ form a 3-parameter Lie group, with a possible parameter vector $\vec{\boldsymbol{\alpha}}(\tau)$ given by the components (with respect to some given basis) of the image $\tau(\vec{\mathbf{0}})$ of the origin $\vec{\mathbf{0}}$.

Remark. It is usual (but by no means necessary) to associate the parameter vector $\vec{\mathbf{0}} = (0, \ldots, 0)$ to the identity element; whenever possible, we shall use the convention $\vec{\boldsymbol{\alpha}}(\mathbf{1}_G) = \vec{\mathbf{0}}$.

Given a local chart $\boldsymbol{\alpha}_W : W \to \mathcal{U}$ in a neighborhood $W \subseteq G$ of the identity element $\mathbf{1}_G$, the group structure implies the existence of continuous maps

$\boldsymbol{\mu} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ and $\boldsymbol{\iota} : \mathcal{U} \to \mathcal{U}$ (the structure functions) such that

$$\vec{\boldsymbol{\alpha}}(gh) = \boldsymbol{\mu}\big(\vec{\boldsymbol{\alpha}}(g)\,, \vec{\boldsymbol{\alpha}}(h)\big)$$
$$\vec{\boldsymbol{\alpha}}\big(g^{-1}\big) = \boldsymbol{\iota}(\vec{\boldsymbol{\alpha}}(g))$$

$\boldsymbol{\mu}\big(\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}\big)$ is the parameter vector of the product of the group elements with parameter vectors $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$, while $\boldsymbol{\iota}(\vec{\boldsymbol{\alpha}})$ is that of the inverse.

Associativity of the group product implies the relation

$$\boldsymbol{\mu}\big(\vec{\boldsymbol{\alpha}}, \boldsymbol{\mu}\big(\vec{\boldsymbol{\beta}}, \vec{\boldsymbol{\gamma}}\big)\big) = \boldsymbol{\mu}\big(\boldsymbol{\mu}\big(\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}\big), \vec{\boldsymbol{\gamma}}\big)$$

for $\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}, \vec{\boldsymbol{\gamma}} \in \mathcal{U}$.

Since $\vec{\boldsymbol{\alpha}}(\mathbf{1}_G)=\vec{\mathbf{0}}$, the other group axioms (existence of the identity and of inverses) take the form

$$\boldsymbol{\mu}(\vec{\boldsymbol{\alpha}},\vec{\mathbf{0}}) = \boldsymbol{\mu}(\vec{\mathbf{0}},\vec{\boldsymbol{\alpha}}) = \vec{\boldsymbol{\alpha}}$$
$$\boldsymbol{\mu}(\vec{\boldsymbol{\alpha}},\boldsymbol{\iota}(\vec{\boldsymbol{\alpha}})) = \boldsymbol{\mu}(\boldsymbol{\iota}(\vec{\boldsymbol{\alpha}}),\vec{\boldsymbol{\alpha}}) = \vec{\mathbf{0}}$$

Remark. Since $\mathcal{U} \subseteq \mathbb{R}^n$, the maps $\boldsymbol{\mu}$ and $\boldsymbol{\iota}$ that characterize locally the group structure can be studied by means of calculus in several variables.

Gleason-Montgomery-Zippin: the structure functions $\boldsymbol{\mu}:\mathcal{U}\times\mathcal{U}\to\mathcal{U}$ and $\boldsymbol{\iota}:\mathcal{U}\to\mathcal{U}$ are analytic, i.e. their Taylor-series around the origin have a positive radius of convergence, if they are twice differentiable.

Examples of Lie groups

1. The additive group $(\mathbb{R}, +)$ of real numbers is a one parameter Lie group, with structure functions $\boldsymbol{\mu}(\alpha, \beta) = \alpha + \beta$ and $\boldsymbol{\iota}(\alpha) = -\alpha$ in case of the trivial parametrization $\boldsymbol{\alpha}(z) = z$.

2. The multiplicative group $\mathsf{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ of complex phases is again a one parameter Lie group. When using the exponential parametrization $\boldsymbol{\alpha}(z) = -\mathsf{i} \log z$, its structure functions are as above: $\boldsymbol{\mu}(\alpha, \beta) = \alpha + \beta$ and $\boldsymbol{\iota}(\alpha) = -\alpha$.

3. The isospin group $\mathsf{SU}(2) = \left\{U \in \mathrm{Mat}_2(\mathbb{C}) \mid \det U = 1,\ U^\dagger U = 1\right\}$ is a three parameter Lie group.

4. The 3D rotation group of rotations around axes having a common point (the center) is a three parameter Lie group (rotations can be parametrized by e.g. the 3 Euler angles). Topologically, it is homeomorphic with the interior of a ball, with identification of some boundary points. Since rotations preserve orientation and the distance from the center, the rotation group can be identified with the group $\mathsf{SO}_3(\mathbb{R})$ of 3-by-3 orthogonal matrices.

5. The Poincaré group $\mathcal{P}$, i.e. the symmetry group of 4D Minkowski space-time, is 10 dimensional Lie group. Four parameters correspond to space-time translations, while 6 to 4D rotations, out of which 3 describe 3D rotations, and another 3 the Lorentz boosts.

Every Lie group G has a canonical parametrization such that

$$\boldsymbol{\mu}(\vec{\boldsymbol{\alpha}},\vec{\boldsymbol{\beta}})_i = \alpha_i + \beta_i + \frac{1}{2}\sum_{j,k=1}^{n} c_i^{jk}\alpha_j\beta_k + \text{higher order terms}$$

and $\boldsymbol{\iota}(\vec{\boldsymbol{\alpha}}) = -\vec{\boldsymbol{\alpha}}$.

Lie's theorem: the coefficients c_k^{ij} satisfy

$$c_i^{jk} + c_i^{kj} = 0 \qquad \text{skew-symmetry}$$

$$\sum_m \left\{ c_i^{jm} c_m^{kl} + c_i^{km} c_m^{lj} + c_i^{lm} c_m^{jk} \right\} = 0 \qquad \text{Jacobi identity}$$

and determine all the higher order terms of the expansion. Moreover, any system of real coefficients c_k^{ij} that satisfy the above requirements corresponds to some Lie group.

The coefficients c_k^{ij}, the so-called structure constants of G, characterize the algebraic structure locally (i.e. near the identity element).

A Lie homomorphism $\phi\colon G_1 \to G_2$ between the Lie groups G_1 and G_2 is a continuous (analytic) group homomorphism, while a local isomorphism is an analytic map that is a bijective homomorphism when restricted to a suitable neighborhood of the identity.

While not necessarily isomorphic, locally isomorphic Lie groups have identical structure functions in suitable parametrizations, hence look the same in some neighborhood of the identity.

A one-parameter subgroup of a Lie group G is a homomorphic image (inside G) of the additive group $(\mathbb{R}, +)$ of real numbers.

2 Lie algebras

A Lie algebra is a linear space $\mathcal{L}$ endowed with a binary operation, the Lie bracket $[,]:\mathcal{L}\times\mathcal{L}\to\mathcal{L}$, that is

- bilinear, i.e. $[\lambda a+\mu b, c]=\lambda[a,c]+\mu[b,c]$ and $[a,\lambda b+\mu c]=\lambda[a,b]+\mu[a,c]$;
- skew-symmetric, i.e. $[b,a]=-[a,b]$;
- satisfies the Jacobi identity

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$$

for all $a,b,c\in\mathcal{L}$ and arbitrary scalars λ,μ.

A Lie algebra homomorphism is a linear map $\phi:\mathcal{L}_1\to\mathcal{L}_2$ such that

$$\phi\big([a,b]\big)=[\phi(a),\phi(b)]$$

Examples:

1. $\mathbb{R}^3$ with the cross product as Lie bracket;
2. $\mathrm{Mat}_n(\mathbb{R})$ with the commutator $[A,B] = AB - BA$ as Lie bracket;
3. the general linear algebra $\mathfrak{gl}(V)$ of all linear operators $A\colon V \to V$ with the commutator $[A,B] = AB - BA$ as Lie bracket;
4. continuous functions on the phase space ('observable quantities') of a Hamiltonian system, with the Poisson bracket as Lie bracket;
5. angular momentum operators in quantum mechanics.

Given a basis $\mathbf{B} = \{b_1, \ldots, b_n\}$ of $\mathcal{L}$, the Lie brackets

$$[b_i, b_j] = \sum_{k=1}^{n} c_k^{ij} b_k$$

of the basis vectors determine the algebra because of bilinearity

$$\left[\sum_i x_i b_i, \sum_j y_j b_j\right] = \sum_k \left(\sum_{i,j} c_k^{ij} x_i y_j\right) b_k$$

The coefficients c_k^{ij}, the structure constants of $\mathcal{L}$, satisfy

$$c_i^{jk} + c_i^{kj} = 0 \qquad \text{skew-symmetry}$$

$$\sum_m \{c_i^{jm} c_m^{kl} + c_i^{km} c_m^{lj} + c_i^{lm} c_m^{jk}\} = 0 \qquad \text{Jacobi identity}$$

and characterize the Lie algebra up to isomorphism.

The structure constants of a Lie group satisfy the same identities $\rightsquigarrow$

correspondence between Lie groups and Lie algebras!

Can reduce questions about Lie groups to questions about Lie algebras.

Advantage: linear structure of Lie algebras!

For example, for a given $x \in \mathcal{L}$ let's consider the mapping

$$\begin{aligned} \mathtt{ad}_x : \mathcal{L} &\to \mathcal{L} \\ y &\mapsto [x, y] \end{aligned}$$

By bilinearity of the Lie bracket, $\mathtt{ad}_x$ is a linear operator on $\mathcal{L}$, whose properties may be described by linear algebraic means (spectral theory, determinants, etc.).

Question: how can we compute the Lie algebra of a Lie group?

Effective methods for Lie transformation groups, i.e. continuous groups of differentiable coordinate transformations $x_i \mapsto x'_i(x_1, \ldots, x_m | \vec{\boldsymbol{\alpha}})$ of $\mathbb{R}^m$, where $\vec{\boldsymbol{\alpha}} \in \mathcal{U} \subseteq \mathbb{R}^n$.

The first order partial differential operators (for $i = 1, \ldots, n$)

$$T_i = \sum_{j=1}^{m} \left(\frac{\partial x'_j}{\partial \alpha_i} \right)_{\vec{\boldsymbol{\alpha}} = \vec{\mathbf{0}}} \frac{\partial}{\partial x_j}$$

(the infinitesimal generators) have commutators

$$[T_i, T_j] = T_i \circ T_j - T_j \circ T_i = \sum_{k=1}^{n} c_k^{ij} T_k$$

with c_k^{ij} the structure constants of the Lie algebra of the group.

Examples

1. The group of 3D translations

$$\begin{aligned} \mathfrak{t}_{\vec{\boldsymbol{\alpha}}} : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \vec{\boldsymbol{x}} &\mapsto \vec{\boldsymbol{x}} + \vec{\boldsymbol{\alpha}} \end{aligned}$$

with $\vec{\boldsymbol{\alpha}} \in \mathbb{R}^3$. The infinitesimal generators read

$$T_i = \sum_j \frac{\partial(x_j + \alpha_j)}{\partial \alpha_i} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_i}$$

Since mixed partial derivatives of differentiable functions are equal, the commutators (Lie brackets) vanish: $[T_i, T_j] = 0$.

More generally, the Lie group of translations of n-dimensional Euclidean space $\mathbb{R}^n$ has n infinitesimal generators whose Lie brackets vanish.

2. Consider the group of all rotations in 2D around the origin. This is a one-parameter Lie group, and the 2D rotation by angle $\alpha \in [0, 2\pi)$ acts on Cartesian coordinates as

$$R(\alpha): \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha\,\mathsf{x} - \sin\alpha\,\mathsf{y} \\ \sin\alpha\,\mathsf{x} + \cos\alpha\,\mathsf{y} \end{pmatrix}$$

There is only one infinitesimal generator, which reads

$$T = \frac{\partial(\cos\alpha\,\mathsf{x} - \sin\alpha\,\mathsf{y})}{\partial\alpha}\frac{\partial}{\partial\mathsf{x}} + \frac{\partial(\sin\alpha\,\mathsf{x} + \cos\alpha\,\mathsf{y})}{\partial\alpha}\frac{\partial}{\partial\mathsf{y}} = -\mathsf{y}\frac{\partial}{\partial\mathsf{x}} + \mathsf{x}\frac{\partial}{\partial\mathsf{y}}$$

Clearly, $[T, T] = 0$ by skew-symmetry of the Lie bracket, which is exactly what one would obtain for translations in 1D, hence the corresponding Lie groups are locally isomorphic.

3. The affine group Aff_n consists of the transformations of $\mathbb{R}^n$

$$\mathfrak{a}(\lambda, \vec{\boldsymbol{a}}) : \vec{\boldsymbol{x}} \mapsto \lambda \vec{\boldsymbol{x}} + \vec{\boldsymbol{a}}$$

with $\vec{\boldsymbol{a}} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The generators read

$$D = \sum_{i=1}^{n} \left(\frac{\partial(\lambda x_i + a_i)}{\partial \lambda} \right)_{\lambda=1, \vec{\boldsymbol{a}}=\vec{\mathbf{0}}} \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

$$T_j = \sum_{i=1}^{n} \left(\frac{\partial(\lambda x_i + a_i)}{\partial a_j} \right)_{\lambda=1, \vec{\boldsymbol{a}}=\vec{\mathbf{0}}} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_j}$$

and one has $[T_i, T_j] = 0$ and

$$[D, T_j] = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right) = -\frac{\partial}{\partial x_j} = -T_j$$

3 Global properties

The Lie algebra **Lie**(G) reflects only the local structure of the Lie group G (in a suitable neighborhood of the identity), the global structure is captured by topology.

Fundamental topological properties:

compactness, if every open covering contains a finite subcovering;

connectedness, if any two group elements may be connected by a continuous curve;

simply connectedness, if any closed curve can be deformed continuously to a point.

In every Lie group, the endpoints of all continuous curves starting at the identity element form a connected subgroup $G_0 < G$, the component of the identity, and there is a one-to-one correspondence between the cosets of G_0 in G and the connected components of G.

Every connected Lie group G is locally isomorphic with a unique simply connected Lie group $\hat{G}$, its universal cover, and there is a discrete central subgroup $Z < Z(\hat{G})$ such that $G \cong \hat{G}/Z$.

Every Lie algebra corresponds to a unique (up to isomorphism)simply connected Lie group, hence the study of Lie algebras parallels that of simply connected groups.

Examples

1. $(\mathbb{R}, +)$ and $\mathsf{U}(1)$ are locally isomorphic, but $(\mathbb{R}, +)$ is simply connected and non-compact, while $\mathsf{U}(1)$ is compact and connected, but not simply connected $\rightsquigarrow$ $(\mathbb{R}, +)$ is the universal cover of $\mathsf{U}(1)$.

2. $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ have the same Lie algebra, hence they are locally isomorphic, but while the former is simply connected and compact, the latter is compact and connected, but not simply connected $\rightsquigarrow$ $\mathsf{SU}(2)$ is the universal cover of $\mathsf{SO}(3)$.

3. the Poincaré group is neither connected (reflections!) nor compact (translations!).

4 The Haar measure

On several occasions one needs to average real-valued functions $f : G \to \mathbb{R}$ over the elements of a group G.

If G is finite, then

$$\langle f \rangle = \frac{1}{|G|} \sum_{g \in G} f(g)$$

If G is continuous, then

$$\langle f \rangle = \frac{1}{\mathtt{vol}(G)} \int f(g)\, \mathtt{d}\mu$$

with the integral taken with respect to a suitable Lebesgue-measure μ, and $\mathtt{vol}(G) = \int 1 \mathtt{d}\mu$, the integral of the constant 1, is the 'volume' of G.

Compatibility with group structure: translation invariance.

For finite G
$$\langle f\rangle = \frac{1}{|G|}\sum_{h\in G} f(h) = \frac{1}{|G|}\sum_{h\in G} f(gh)$$
Should hold for topological groups as well!

An invariant measure μ on a topological group G is a Lebesgue-measure such that for every $g\in G$ and every measurable set $U\subseteq G$ the translate $gU=\{gx \mid x\in U\}$ is also measurable, and
$$\mu(gU) = \mu(U)$$

Haar's theorem: every compact topological group admits an invariant measure, the Haar measure, unique up to normalization.

5 The 3D rotation group

Consider the group of 3D rotations around axes having a point in common (the rotation center).

Because rotations transform Cartesian coordinates linearly, they form a Lie transformation group. Choosing Cartesian coordinates $\mathsf{x}, \mathsf{y}, \mathsf{z}$ (origin at the rotation center), the transformed coordinates read

$$\begin{pmatrix} \mathsf{x}' \\ \mathsf{y}' \\ \mathsf{z}' \end{pmatrix} = \mathcal{O} \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \\ \mathsf{z} \end{pmatrix}$$

for a suitable 3-by-3 matrix $\mathcal{O} \in \mathrm{Mat}_3(\mathbb{R})$.

Since rotations preserve orientation one has $\det \mathcal{O} > 0$, and because the distance $\sqrt{x^2+y^2+z^2}$ from the origin (rotation center) is invariant, $\mathcal{O}$ is an orthogonal matrix, i.e. $\mathcal{O}^{tr}\mathcal{O} = \mathbf{1}_3$, hence there exists a one-to-one correspondence between 3D rotations and 3-by-3 orthogonal matrices whose determinant equals 1.

The group of 3D rotations is isomorphic with the matrix group $SO(3)$.

Each rotation is characterized by its rotation angle and the direction of its rotation axis, hence one needs 3 angular coordinates to parametrize 3D rotations (e.g. the Euler angles): the rotation group is a 3-parameter Lie group.

Any rotation can be decomposed into a product of three consecutive rotations around perpendicular axes:

$$\mathcal{O}(\vec{\boldsymbol{\alpha}}) = \mathcal{O}_x(\alpha_x)\,\mathcal{O}_y(\alpha_y)\,\mathcal{O}_z(\alpha_z)$$

where

$$\mathcal{O}_z(\alpha): \begin{pmatrix} \mathsf{x} \\ \mathsf{y} \\ \mathsf{z} \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha\,\mathsf{x} - \sin\alpha\,\mathsf{y} \\ \sin\alpha\,\mathsf{x} + \cos\alpha\,\mathsf{y} \\ \mathsf{z} \end{pmatrix}$$

with infinitesimal generator

$$L_z = \mathsf{x}\frac{\partial}{\partial \mathsf{y}} - \mathsf{y}\frac{\partial}{\partial \mathsf{x}}$$

Similarly, one has

$$L_x = \mathsf{y}\frac{\partial}{\partial \mathsf{z}} - \mathsf{z}\frac{\partial}{\partial \mathsf{y}}$$
$$L_y = \mathsf{z}\frac{\partial}{\partial \mathsf{x}} - \mathsf{x}\frac{\partial}{\partial \mathsf{z}}$$

The Lie-algebra is spanned by (real) linear combinations of L_x, L_y, L_z.

Lie brackets from commutators of infinitesimal generators

$$\begin{aligned}[L_x, L_y] &= \left(\mathsf{y}\frac{\partial}{\partial \mathsf{z}} - \mathsf{z}\frac{\partial}{\partial \mathsf{y}}\right)\left(\mathsf{z}\frac{\partial}{\partial \mathsf{x}} - \mathsf{x}\frac{\partial}{\partial \mathsf{z}}\right) - \left(\mathsf{z}\frac{\partial}{\partial \mathsf{x}} - \mathsf{x}\frac{\partial}{\partial \mathsf{z}}\right)\left(\mathsf{y}\frac{\partial}{\partial \mathsf{z}} - \mathsf{z}\frac{\partial}{\partial \mathsf{y}}\right)\\ &= \mathsf{y}\frac{\partial}{\partial \mathsf{z}}\left(\mathsf{z}\frac{\partial}{\partial \mathsf{x}} - \mathsf{x}\frac{\partial}{\partial \mathsf{z}}\right) - \mathsf{z}\frac{\partial}{\partial \mathsf{y}}\left(\mathsf{z}\frac{\partial}{\partial \mathsf{x}} - \mathsf{x}\frac{\partial}{\partial \mathsf{z}}\right) - \mathsf{z}\frac{\partial}{\partial \mathsf{x}}\left(\mathsf{y}\frac{\partial}{\partial \mathsf{z}} - \mathsf{z}\frac{\partial}{\partial \mathsf{y}}\right)\\ &\qquad + \mathsf{x}\frac{\partial}{\partial \mathsf{z}}\left(\mathsf{y}\frac{\partial}{\partial \mathsf{z}} - \mathsf{z}\frac{\partial}{\partial \mathsf{y}}\right) = \mathsf{y}\frac{\partial}{\partial \mathsf{x}} - \mathsf{x}\frac{\partial}{\partial \mathsf{y}} = -\mathsf{L}_\mathsf{z}\end{aligned}$$

Similarly

$$[L_x, L_z] = L_y$$
$$[L_y, L_z] = -L_x$$

The generator of rotations around an axis parallel to $\vec{\boldsymbol{n}}$ is

$$L_{\vec{\boldsymbol{n}}} = n_x L_x + n_y L_y + n_z L_z$$

and their commutator reads

$$[L_{\vec{\boldsymbol{n}}}, L_{\vec{\boldsymbol{m}}}] = L_{\vec{\boldsymbol{n}} \times \vec{\boldsymbol{m}}}$$

the Lie algebra of the rotation group is isomorphic with that of 3D vectors

(with the cross product as Lie bracket).

Noether's theorem: each 1-parameter group of symmetries of a physical system corresponds to a conserved quantity (with rotational invariance corresponding to the conservation of angular momentum).

Question: is there a relation between angular momentum and infinitesimal generators of the rotation group?

In QM, to any observable quantity corresponds a self-adjoint operator, whose eigenvalues are the possible measurement outcomes.

Guess: the infinitesimal generators are linear operators acting on the Hillbert space $L^2(\mathbb{R}^3)$ of square-integrable functions, i.e. the space of wave functions, hence they could be related to the components of the angular momentum operator.

Problem: the eigenvalues of the infinitesimal generators are dimensionless quantities, unless those of the angular momentum operator, hence one needs to rescale them by a quantity with the dimension of angular momentum: a natural choice is Planck's constant $\hbar$.

Still not enough, because the infinitesimal generators are anti-hermitian operators on $L^2(\mathbb{R}^3)$

$$\langle f, L_i g\rangle = \int \overline{f(\mathsf{x},\mathsf{y},\mathsf{z})} L_i g(\mathsf{x},\mathsf{y},\mathsf{z})\, \mathsf{dxdydz} = -\langle L_i f, g\rangle$$

hence their eigenvalues are not real, but purely imaginary numbers (whose square is negative), consequently they cannot correspond to 'observable quantities' (whose eigenvalues are real numbers).

Solution: multiply the infinitesimal generators L_i with the imaginary number $-\mathrm{i}\,\hbar$.

The operators $J_i = -\mathrm{i}\,\hbar L_i$ are self-adjoint , and their Lie brackets (ϵ_{ijk} is the Levi-Civita tensor)

$$[J_i, J_j] = \mathrm{i}\,\epsilon_{ijk}\hbar J_k$$

reproduce the commutation rules of the components of the angular momentum operator!

Note that the J_i do not belong to the Lie algebra properly, only to its so-called complexification.

The Lie algebra $\mathfrak{su}(2)$ of the isospin group $\mathsf{SU}(2)$ consists of the traceless and self-adjoint 2-by-2 matrices

$$\mathfrak{su}(2) = \left\{ A \in \mathrm{Mat}_2\,(\mathbb{C}) \mid A^\dagger = A \text{ and } \mathrm{Tr}(A) = 0 \right\}$$

with Lie bracket the commutator of matrices.

A basis of $\mathfrak{su}(2)$ is provided by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & \mathfrak{i} \\ -\mathfrak{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

whose commutators read

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = \frac{\sigma_i}{2}\frac{\sigma_j}{2} - \frac{\sigma_j}{2}\frac{\sigma_i}{2} = \mathfrak{i}\,\epsilon_{ijk}\frac{\sigma_k}{2}$$

The matrices $\frac{\hbar}{2}\sigma_i$ have the same Lie brackets as the generators J_i, hence $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ have isomorphic Lie algebras, i.e. they are locally isomorphic. Since $\mathsf{SU}(2)$ is simply connected, it is the universal cover of $\mathsf{SO}(3)$, hence the latter is a factor group by a central subgroup. Because $\mathsf{SU}(2)$ is not isomorphic with $\mathsf{SO}(3)$ (while being locally isomorphic to it), the central subgroup cannot be trivial. As the center Z of $\mathsf{SU}(2)$ consists of the two matrices

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the only non-trivial central subgroup of $\mathsf{SU}(2)$ is Z itself, from which we conclude that

$$\mathsf{SO}(3) \cong \mathsf{SU}(2)/\mathsf{Z}$$