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The perturbative structure of spin glass field theory

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Abstract

Cubic replicated field theory is used to study the glassy phase of the short-range Ising spin glass just below the transition temperature, and for systems above, at, and slightly below the upper critical dimension six. The order parameter function is computed up to two-loop order. There are two, well-separated bands in the mass spectrum, just as in mean field theory. The small mass band acts as an infrared cutoff, whereas contributions from the large mass region can be computed perturbatively ($d > 6$), or interpreted by the ϵ -expansion around the critical fixed point ($d = 6 - \epsilon$). The one-loop calculation of the (momentum-dependent) longitudinal mass, and the whole replicon sector is also presented. The innocuous behavior of the replicon masses while crossing the upper critical dimension shows that the ultrametric replica symmetry broken phase remains stable below six dimensions.

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1. Introduction

A spin glass is a prototype of complex systems, with its slow dynamics on macroscopic time scales, unusual equilibrium properties, and complicated phase space structure which breaks ergodicity. The interest in the understanding of the spin glass problem started in the seventies of the last century, and lots of results have accumulated since then, nevertheless many basic questions have remained open. (For an overview of the history of spin glass research, see review papers from different periods: [1–4].)

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Numerical simulations are important tools for getting information about spin glass properties. Without trying to overview this huge field, we only mention here the Janus Collaboration using the special purpose Janus computer, providing results about the spin glass phase in the physical three-dimensional Edwards–Anderson model which are compatible with an ultrametrically organized replica symmetry broken (RSB) phase (see [5] for a recent list of references related to the Janus Collaboration). This ultrametric glassy phase emerged for the first time in the solution of the mean field theory of the Ising spin glass by Parisi, see [2] and references therein, and — according to our present knowledge — it seems to persist below the upper critical dimension six [6]; possibly (as the aforementioned numerical simulations suggest) down to three dimensions. Nevertheless the details of the RSB phase of the short range finite-dimensional model differ in many ways from its mean field counterparts. Important examples for such discrepancies are the leading behavior of the order parameter function and momentum-dependent masses (or, equivalently, correlation functions) close to criticality. Moreover, these details depend on the space dimension d which can be well illustrated by the breakpoint x_1 of the order parameter function $q(x)$, see Refs. [6–9]: x_1 is proportional to $\tau \sim (T_c - T)/T_c$ in mean field theory (which is equivalent to the infinite-dimensional model), and this behavior persists down to $d = 8$, with possibly a logarithm of τ at exactly eight dimensions. For $6 < d < 8$, $x_1 \sim \tau^{\frac{4}{2}-3}$, whereas at exactly six dimensions: $x_1 \sim |\ln \tau|^{-1}$. Below six dimensions x_1 becomes finite at T_c , and renormalization group arguments show [6] that its critical value is universal. It was computed in first order in $\epsilon = 6 - d$ in Ref. [6], the present paper extends this calculation to second order, see Eq. (31). There is a trend of increasing x_1 with decreasing d , a clear sign that RSB becomes more dominant. This contradicts expectations that a replica symmetric (RS) glassy phase, which is characterized by the so-called “droplet” picture [10–12], enters in some low dimension, for which a possible scenario would be if x_1 decreased to zero. Well below $d = 6$ one expects x_1 to be of order unity, meaning that intervalley overlaps become as important as self-overlap [2].

As an alternative to numerical simulations, replicated field theory provides analytic results, and it has the advantage that space dimension can be chosen at will by defining the model on a d -dimensional hypercubic lattice. In the present paper, spin glass field theory is studied below eight dimensions, also passing through the upper critical dimension six. (In this domain of dimensions, a simple cubic model with the coupling constant w is sufficient for obtaining critically relevant properties, the quartic coupling of the truncated model, for instance, which is dangerously irrelevant in higher dimensions, can now be neglected.) Our main purpose is to understand how the perturbative method works in this system whose mass spectrum consists of two separated bands: a large one dominating the behavior in the “near infrared” momentum range, and a small one — extending to zero — related to the “far infrared” sector. We extend former calculations of the order parameter function $q(x)$ to two-loop order. This calculation needs computing some of the one-loop self-energy insertions; the results can be used to get the momentum-dependent longitudinal mass and the replicon band in one-loop order. The findings for the replicon band support the idea that stability of the RSB phase persists below six dimensions.

The paper is organized as follows: The model is defined and the equation of state for the order parameter function presented in Section 2, while the properties of the free propagators are studied in Section 3. The different terms contributing to the equation of state up to two-loop order are worked out in Section 4. Section 5 contains the main results for the order parameter function in the different dimensional regimes, namely $6 < d < 8$, $d = 6$, and $d \lesssim 6$. This section is divided into three sections: Section 5.1 is for the breakpoint x_1 , Section 5.2 for the Edwards–Anderson order parameter, while Section 5.3 is devoted to $q(x)$ with $x < x_1$. The momentum-dependent mass (the inverse propagator) is studied in Section 6. In Section 6.1 the longitudinal mass is

computed, and its scaling behavior below six dimensions displayed and proved. The replicon sector is left to Section 6.2. The discussion of the results is included in the last section, i.e. in Section 7. Several results are summarized in listed and tabulated forms in the three appendices.

2. The replicated cubic field theory and mass renormalization

The replicated field theory (representing the Ising spin glass on a d -dimensional hypercubic lattice in zero external magnetic field) has its dynamical variables $\phi^{\alpha\beta} = \phi^{\beta\alpha}$ — with $\phi^{\alpha\alpha} = 0$ and replica indices α, β, \dots taking values from 1 to n where n is the replica number — and Lagrangian $\mathcal{L}(\phi^{\alpha\beta})$ which is invariant under any permutations of the replicas (a trivial outcome of the replica trick) and under the transformation $\phi'^{\alpha\beta} = (-1)^{\alpha+\beta} \phi^{\alpha\beta}$ (expressing the extra symmetry coming with the vanishing external field [13]). Concentrating on the glassy phase just below the critical temperature, invariants of the above symmetry which are higher than cubic can be neglected when $d < 8$, and the Lagrangian takes the relatively simple form

$$\mathcal{L} = \frac{1}{2} \sum_{\mathbf{p}} \left(\frac{1}{2} p^2 + m_c - \tau \right) \sum_{\alpha\beta} \phi_{\mathbf{p}}^{\alpha\beta} \phi_{-\mathbf{p}}^{\alpha\beta} - \frac{1}{6N^{1/2}} w \sum' \sum_{\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\beta\gamma} \phi_{\mathbf{p}_3}^{\gamma\alpha}. \quad (1)$$

Momentum conservation is indicated by the primed summation. The number N of the Ising spins becomes infinite in the thermodynamic limit, rendering summations to integrals over the continuum of momenta in the diagrams of the perturbative expansion. A momentum cutoff Λ is always understood to block ultraviolet divergences. The two important parameters of the model are the reduced temperature τ and the coupling constant w . As we are working in the immediate vicinity of the critical point, τ is assumed to be much smaller than Λ^2 . The critical bare mass can be computed relatively easily yielding (up to second order and for $n = 0$):

$$\begin{aligned} m_c &= m_c^{(1)} + m_c^{(2)} \\ &= -w^2 \frac{1}{N} \sum_{\mathbf{p}} \frac{1}{p^4} + w^4 \frac{1}{N^2} \sum_{\mathbf{p}, \mathbf{q}} \left\{ \frac{4}{p^6 q^2} \left[\frac{1}{(\mathbf{p} - \mathbf{q})^2} - \frac{1}{q^2} \right] \right. \\ &\quad \left. + \frac{1}{p^4 q^2 (\mathbf{p} - \mathbf{q})^2} \left[\frac{1}{q^2} + \frac{1}{(\mathbf{p} - \mathbf{q})^2} \right] \right\}. \end{aligned} \quad (2)$$

For studying the glassy phase, the replica symmetric Lagrangian above is converted to the replica symmetry broken one by the transformation $\phi_{\mathbf{p}}^{\alpha\beta} \rightarrow \phi_{\mathbf{p}}^{\alpha\beta} - \sqrt{N} q_{\alpha\beta} \delta_{\mathbf{p}=0}^{\text{Kr}}$ where $q_{\alpha\beta} \equiv \langle \phi_i^{\alpha\beta} \rangle$ is the *exact* homogeneous order parameter matrix. The Lagrangian in (1) gets then the additional mass term $-\frac{1}{2} w \sum_{\mathbf{p}} \sum_{\alpha\beta\gamma} q_{\alpha\beta} \phi_{\mathbf{p}}^{\beta\gamma} \phi_{-\mathbf{p}}^{\gamma\alpha}$. The new fields have, by definition, zero mean now, and this condition yields the equation of state for $q_{\alpha\beta}$:

$$2\tau q_{\alpha\beta} + w(q^2)_{\alpha\beta} + w \frac{1}{N} \sum_{\mathbf{p}} \sum_{\gamma \neq \alpha, \beta} G_{\alpha\gamma, \beta\gamma}^{\text{exact}} - 2m_c q_{\alpha\beta} = 0, \quad (3)$$

where the exact propagator satisfies Dyson's equation:

$$(G^{\text{exact}})^{-1} = p^2 + M - \Sigma \quad \text{with} \quad \begin{cases} M_{\alpha\beta, \alpha\beta} = -2\tau + 2m_c, \\ M_{\alpha\gamma, \beta\gamma} = -w q_{\alpha\beta}, \\ M_{\alpha\beta, \gamma\delta} = 0, \end{cases} \quad \text{and } \Sigma \text{ the self-energy.} \quad (4)$$

From now on, n is set to zero (the spin glass limit), and an infinite-step ultrametric structure [2] is assumed for $q_{\alpha\beta} = q(x)$, with the overlap $x = \alpha \cap \beta$. Construction of the free propagator G in

spin glass field theory is not a trivial task. One can be guided by lessons learnt from Refs. [7–9]: $\tau = wq_1 + \dots$ and $q(x) \sim x$ in leading order. To generate a perturbation theory where the small parameter is the breakpoint x_1 of the order parameter function, one can divide the mass as $M = M_0 + M_1$, and q as $q(x) = q_1 \bar{q}(r) = q_1 [r + \Delta \bar{q}(r)]$ with $r \equiv x/x_1$. By definition, $\Delta \bar{q}(1) = 0$ — since q_1 and x_1 are exact quantities —, and for the free propagator we have $G^{-1} \equiv p^2 + M_0$ with

$$\begin{aligned} (M_0)_{\alpha\beta,\alpha\beta} &= -2wq_1(1 - x_1/2), \\ (M_0)_{\alpha\gamma,\beta\gamma} &= -wq_1r, \quad \text{with } r = x/x_1 \text{ and } x = \alpha \cap \beta, \\ (M_0)_{\alpha\beta,\gamma\delta} &= 0. \end{aligned} \tag{5}$$

M_1 , on the other hand, plays the role of a quadratic counter term, and the exact propagator has the following expansion:

$$G^{\text{exact}} = G + G(\Sigma - M_1)G + \dots \tag{6}$$

with

$$(M_1)_{\alpha\beta,\alpha\beta} = -2\tau + 2wq_1 + 2m_c - x_1wq_1 \equiv \delta M \quad \text{and} \quad (M_1)_{\alpha\gamma,\beta\gamma} = -wq_1 \Delta \bar{q}(r). \tag{7}$$

3. The free propagator

One can follow Ref. [8] step by step to construct the free propagator from M_0 in Eq. (5). The replicon masses are easily found and can be displayed in the usual parametrization as

$$\lambda_0(x; u, v) = \frac{1}{2}x_1wq_1 \left[\left(\frac{u}{x_1} \right)^2 + \left(\frac{v}{x_1} \right)^2 \right], \quad 0 \leq x \leq u, v \leq x_1. \tag{8}$$

Due to the second term of the diagonal element in Eq. (5), this replicon band extends from zero to x_1wq_1 , and it is necessary for having a positive mass spectrum. While the replicon eigenvalues are exact and of order $\sim x_1wq_1$, the band of large masses is centered around $2wq_1$ and has the expansion:

$$\lambda_0(k) = 2wq_1 \left\{ 1 + \frac{1}{6} \left[2 \left(\frac{k}{x_1} \right)^3 - 1 \right] x_1 + O(x_1^2) \right\}, \quad 0 \leq k \leq x_1.$$

One can see that the small parameter x_1 has a double role in the mass spectrum: Firstly, it is the ratio of the small to large masses which are then clearly separated and, secondly, the widths of both bands are proportional to x_1 .

Inversion of the mass operator is complicated, but feasible by the techniques of Ref. [8]. As an illustration of the behavior of the free propagator components as a function of the momentum, let us consider the combination entering the equation of state (3)¹:

$$Y_{\alpha\beta} \equiv Y(x) \equiv \sum_{\gamma \neq \alpha, \beta} G_{\alpha\gamma,\beta\gamma} = - \int_0^x dy G_{1x}^{yy} - xG_1^{xx} - 2 \int_x^1 dy G_1^{xy}, \quad x = \alpha \cap \beta. \tag{9}$$

Y has the following expansion in the large (l) and small (s) mass regimes:

¹ For the parametrization of the components of an ultrametric matrix, see Ref. [8].

- Large mass regime, i.e. $p^2 \sim 2wq_1$:

$$Y = \frac{1}{p^2} [G_0^{(l)}(u; r) + x_1 G_1^{(l)}(u; r) + x_1^2 G_2^{(l)}(u; r) + \dots], \quad u = \frac{p^2}{2wq_1} \text{ and } r = \frac{x}{x_1}.$$

When $p^2 < 2wq_1$, we have the following expansion of the $G^{(l)}$ functions (although not indicated, the $g^{(l)}$ coefficients are still functions of r):

$$\begin{aligned} G_0^{(l)}(u) &= u^{-1}(g_{00}^{(l)} + g_{01}^{(l)}u + g_{02}^{(l)}u^2 + \dots), \\ G_1^{(l)}(u) &= u^{-2}(g_{10}^{(l)} + g_{11}^{(l)}u + \dots), \\ G_2^{(l)}(u) &= u^{-3}(g_{20}^{(l)} + \dots), \dots \end{aligned}$$

- Small mass regime, i.e. $p^2 \sim x_1 \times 2wq_1$:

$$Y = \frac{1}{x_1 p^2} [G_0^{(s)}(u; r) + x_1 G_1^{(s)}(u; r) + x_1^2 G_2^{(s)}(u; r) + \dots],$$

$$u = \frac{p^2}{x_1 2wq_1} \text{ and } r = \frac{x}{x_1}.$$

The expansion of the $G^{(s)}$ functions for $p^2 > x_1 2wq_1$ (the r -dependence of the coefficients not indicated):

$$\begin{aligned} G_0^{(s)}(u) &= u^{-1}(g_{00}^{(s)} + g_{01}^{(s)}u^{-1} + g_{02}^{(s)}u^{-2} + \dots), \\ G_1^{(s)}(u) &= (g_{10}^{(s)} + g_{11}^{(s)}u^{-1} + \dots), \\ G_2^{(s)}(u) &= u(g_{20}^{(s)} + \dots), \\ &\dots \end{aligned}$$

It is $G_0^{(l)}$ which can be computed by the least effort using the “smallest block approximation” (SBA), meaning that the γ summation in (9) is restricted to the smallest ultrametric blocks of size x_1 around α and β :

$$Y \approx 2(x_1 - 1)G_1^{x_1} \approx -2G_1^{x_1} \implies G_0^{(l)}(u; r) = \frac{-(u^2 + 3u + 1)r + r^3}{u(u + 1)^2} \tag{10}$$

where the near infrared form of $G_1^{x_1}$ from Section 6 of Ref. [8] has been used. The calculation of $G_1^{(l)}$ — which is necessary for obtaining the corrections to the order parameter function — is somewhat complicated, and the result is not displayed here [but Eq. (17) is based on that calculation].

There is complete matching in the momentum range $x_1 2wq_1 < p^2 < 2wq_1$ which is satisfied by the following matching-conditions between the two types of coefficients:

$$g_{00}^{(l)} = g_{00}^{(s)}; \quad g_{10}^{(l)} = g_{01}^{(s)}, \quad g_{01}^{(l)} = g_{10}^{(s)}; \quad g_{20}^{(l)} = g_{02}^{(s)}, \quad g_{11}^{(l)} = g_{11}^{(s)}, \quad g_{02}^{(l)} = g_{20}^{(s)}; \quad \dots$$

4. Contributions to the equation of state up to two-loop order

In this section, a summary of the results for the different terms appearing in Eq. (3) are presented.

4.1. The zero-loop term

Assuming $q(x) = q_1[r + \Delta\bar{q}(r)]$ and $r = x/x_1$, the first two terms in (3) can be written (for $\alpha \cap \beta = x$) as

$$\begin{aligned} &2\tau q(x) + wq^2(x) \\ &= 2\tau q_1[r + \Delta\bar{q}(r)] \\ &\quad + wq_1^2 \left[-2r + x_1 \left(r - \frac{1}{3}r^3 \right) - 2\Delta\bar{q}(r) + 2x_1 \{r; \Delta\bar{q}(r)\} + x_1 \{ \Delta\bar{q}(r); \Delta\bar{q}(r) \} \right] \end{aligned}$$

with the bilinear form defined by

$$\begin{aligned} \{f(r); g(r)\} \equiv & f(r)g(1) + f(1)g(r) - f(r) \int_r^1 du g(u) - g(r) \int_r^1 du f(u) \\ & - rf(r)g(r) - \int_0^r du f(u)g(u). \end{aligned} \tag{11}$$

The last term which is quadratic in $\Delta\bar{q}$ will be neglected in the present stage of approximation. We will need the following derivatives:

$$\frac{d}{dr} [2\tau q(x) + wq^2(x)]|_{r=1} = q_1(2\tau - 2wq_1)[1 + \Delta\bar{q}'(r=1)], \tag{12a}$$

$$\begin{aligned} \frac{d^2}{dr^2} [2\tau q(x) + wq^2(x)] = & -2x_1wq_1^2r - 4x_1wq_1^2r\Delta\bar{q}' \\ & + [q_1(2\tau - 2wq_1) + x_1wq_1^2(1 - r^2)]\Delta\bar{q}'' . \end{aligned} \tag{12b}$$

4.2. The one-loop term

Inserting the free propagator [i.e. the leading term in (6)] and $m_c^{(1)}$ into (3), and using the definition of Y in (9), we have

$$X_1 \equiv w \frac{1}{N} \sum_{\mathbf{p}} Y(x) - 2m_c^{(1)}q_1r = wq_1^2\bar{X}_1 \tag{13}$$

where the dimensionless quantity \bar{X}_1 depends only on the three dimensionless parameters of the theory, namely $\bar{g} \equiv w^2K_d/\Lambda^\epsilon$ (the dimensionless coupling constant), $\bar{\Lambda} \equiv \Lambda/(2wq_1)^{1/2}$, and x_1 :

$$\bar{X}_1 = \bar{\Lambda}^\epsilon \bar{g} [F_0^{(l)}(\bar{\Lambda}; r) + x_1 F_1^{(l)}(\bar{\Lambda}; r) + x_1^{\frac{d}{2}-2} F_0^{(s)}(r) + \dots]. \tag{14}$$

In the above formula the sub- and superscripts of the F functions correspond to the expansion of the free propagator in Section 3: thus (l) [(s)] refers to the large (small) mass regime, respectively. Introducing the notations

$$G_R \equiv \frac{1}{p^2} \quad \text{and} \quad G_L \equiv \frac{1}{p^2 + 1} \tag{15}$$

for the (dimensionless) “replicon” and “longitudinal” propagators, Eqs. (10) and (2) give $F_0^{(l)}$:

$$F_0^{(l)}(\bar{\Lambda}; r) = 4 \int_0^{\bar{\Lambda}} dp p^{d-1} (-r G_R G_L^2 + r^3 G_R^2 G_L^2). \tag{16}$$

For obtaining $F_1^{(l)}$, one must go beyond the near infrared approximation for $G_1^{xx_1}$, and also SBA for Y . A lengthy calculation gives

$$F_1^{(l)}(\bar{\Lambda}; r) = 4 \int_0^{\bar{\Lambda}} dp p^{d-1} \left[\left(\frac{1}{2}r + \frac{1}{6}r^3 \right) G_R G_L^2 + \left(-\frac{4}{15}r + \frac{7}{6}r^3 \right) G_R^2 G_L^2 \right. \\ \left. + \left(-\frac{1}{15}r - \frac{1}{3}r^3 \right) G_R G_L^3 + \left(-\frac{23}{120}r + \frac{13}{12}r^3 - \frac{57}{40}r^5 \right) G_R^3 G_L^2 \right. \\ \left. + \frac{4}{15}r^6 G_R^3 G_L^3 \right]. \tag{17}$$

Neglecting contributions which are irrelevant close to the critical temperature, one can write:

$$F_1^{(l)}(\bar{\Lambda}; r) = -\frac{2}{3}(3 + r^2)r \frac{1}{\epsilon} \bar{\Lambda}^{-\epsilon} + \bar{\Lambda}\text{-independent constant.}$$

The constant is singular in six dimensions, namely it can be written as

$$\frac{2}{3}(3 + r^2)r \frac{1}{\epsilon} + \left(\frac{23}{30}r - \frac{13}{3}r^3 + \frac{57}{10}r^5 - \frac{16}{15}r^6 \right) \frac{1}{\epsilon} + O(1).$$

Unfortunately, the small mass term $F_0^{(s)}$ is too complicated to get it in closed form. Nevertheless, its singular behavior at $d = 6$ can be extracted, and one gets a well-defined limit of $F_1^{(l)} + x_1^{-\epsilon/2} F_0^{(s)}$ in six dimensions:

$$F_1^{(l)}(\bar{\Lambda}; r) + x_1^{-\epsilon/2} F_0^{(s)}(r) = \frac{2}{3}(3 + r^2)r \ln \bar{\Lambda} + \frac{1}{2} \left(\frac{23}{30}r - \frac{13}{3}r^3 + \frac{57}{10}r^5 - \frac{16}{15}r^6 \right) \ln x_1 + O(1), \quad d = 6. \tag{18}$$

4.3. The one-loop results for the equation of state

The first and second derivatives (with respect to r) of $F_0^{(l)}$ and the leading parts of Eqs. (12a) and (12b), i.e. neglecting $\Delta\bar{q}$, give the equations between τ and q_1 on the one hand:

$$q_1(2\tau - 2wq_1) = 4wq_1^2 \bar{\Lambda}^\epsilon \bar{g} \int_0^{\bar{\Lambda}} dp p^{d-1} [G_R G_L^2 - 3G_R^2 G_L^2], \quad 6 < d < 8, \tag{19a}$$

and between the dimensionless quantities on the other hand, in leading order:

$$x_1 = 12\bar{\Lambda}^\epsilon \bar{g} \int_0^\infty dp p^{d-1} G_R^2 G_L^2 = 6\Gamma\left(\frac{d}{2} - 2\right)\Gamma\left(4 - \frac{d}{2}\right) \bar{\Lambda}^\epsilon \bar{g} \equiv C_d \bar{\Lambda}^\epsilon \bar{g}, \tag{19b}$$

$6 < d < 8.$

The diagonal mass counterterm δM in (7) can now be computed by the help of Eqs. (2), (19a), and (19b):

$$\delta M = -4wq_1 \bar{\Lambda}^\epsilon \bar{g} \int_0^{\bar{\Lambda}} dp p^{d-1} [G_R G_L^2 + G_R^2]. \tag{20}$$

4.4. The $\Delta\bar{q}$ insertion term

Inserting $-GM_1G$ for G^{exact} into Eq. (3) with the off-diagonal part of M_1 which is proportional to $\Delta\bar{q}$, see Eqs. (6) and (7), and applying the SBA (which is correct up to this order) one arrives at:

$$\begin{aligned} X_{\Delta\bar{q}} &\equiv -w \frac{1}{N} \sum_{\mathbf{p}} \sum_{\gamma \neq \alpha, \beta} \sum_{\mu \neq \nu \neq \rho} G_{\alpha\gamma, \mu\rho}(M_1)_{\mu\rho, \nu\rho} G_{\nu\rho, \beta\gamma} - 2m_c^{(1)} q_1 \Delta\bar{q}(r) \\ &= wq_1^2 \Delta\bar{q}(r) 4\bar{\Lambda}^\epsilon \bar{g} \int_0^{\bar{\Lambda}} dp p^{d-1} [-G_R G_L^2 + 3r^2 G_R^2 G_L^2]. \end{aligned}$$

We can now use Eqs. (19a) and (19b) in correction terms containing $\Delta\bar{q}$, and also let $\bar{\Lambda}$ go to infinity whenever possible (thus neglecting irrelevant contributions close to T_c). Somewhat surprisingly, terms with $\Delta\bar{q}'$ and $\Delta\bar{q}''$ both disappear due to exact cancellations when the $\Delta\bar{q}$ insertion term is added to the zero-loop results in Eqs. (12a) and (12b). The following simple results (correct up to the present second order calculation) are obtained:

$$\frac{d}{dr} [2\tau q(x) + wq^2(x) + X_{\Delta\bar{q}}] \Big|_{r=1} = q_1(2\tau - 2wq_1), \tag{21a}$$

and

$$\frac{d^2}{dr^2} [2\tau q(x) + wq^2(x) + X_{\Delta\bar{q}}] = -2wq_1^2 x_1 [r - \Delta\bar{q}(r)]. \tag{21b}$$

4.5. The remaining two-loop term

By the help of (6) and (7), the basic two-loop contribution to the equation of state in (3) can be written as

$$\begin{aligned} X_2 &\equiv w \frac{1}{N} \sum_{\mathbf{p}} \sum_{\gamma \neq \alpha, \beta} (G \Sigma G - \delta M G^2)_{\alpha\gamma, \beta\gamma} - 2m_c^{(2)} q_1 r \\ &= -2w \frac{1}{N} \sum_{\mathbf{p}} (G \Sigma G - \delta M G^2)_1^{xx_1} - 2m_c^{(2)} q_1 r \end{aligned} \tag{22}$$

where the last equation was obtained by the SBA, hereby neglecting terms which are smaller by a factor of x_1 . For making easier to display and analyze the following — rather complicated — formulae, it is useful to define the following special linear combinations of a generic ultrametric matrix (such as G or Σ , the former is used in the definitions below), see also footnote 1:

$$\begin{aligned} G_R^{xx} &\equiv G_{11}^{xx} - 2G_{1x_1}^{xx} + G_{x_1x_1}^{xx}, & G_L^{xx} &\equiv G_{11}^{xx} - 4G_{1x_1}^{xx} + 3G_{x_1x_1}^{xx}, \\ G_{LA}^{xx} &\equiv 2G_{1x_1}^{xx} - 3G_{x_1x_1}^{xx}; & \delta G^{xx_1} &\equiv G_1^{xx_1} - G_{x_1}^{xx_1}. \end{aligned} \tag{23}$$

Evaluating the matrix products by the SBA again, we get:

$$\begin{aligned}
 & (G\Sigma G - \delta MG^2)_1^{xx_1} \\
 &= [G_R^{xx} G_1^{xx_1} + (2G_L^{xx} - G_L^{xx})\delta G^{xx_1}](\Sigma_R^{xx} - \delta M) - G_R^{xx}\delta G^{xx_1}(\Sigma_L^{xx} - \delta M) \\
 &\quad - G_L^{x_1x_1}\delta G^{xx_1}\Sigma_x^{x_1x_1} + G_R^{xx}G_L^{x_1x_1}\Sigma_1^{xx_1} \\
 &\quad + [G_R^{xx}(-G_x^{x_1x_1} + G_L^{x_1x_1} + G_{LA}^{x_1x_1}) - G_L^{xx}G_L^{x_1x_1} - 4(\delta G^{xx_1})^2]\delta\Sigma^{xx_1} \\
 &\quad + (-G_x^{x_1x_1}\delta G^{xx_1} + G_L^{x_1x_1}G_1^{xx_1} + G_{LA}^{x_1x_1}\delta G^{xx_1})(\Sigma_L^{x_1x_1} - \delta M) \\
 &\quad + G_L^{x_1x_1}\delta G^{xx_1}\Sigma_{LA}^{x_1x_1}.
 \end{aligned} \tag{24}$$

The one-loop self-energy, denoted here simply by Σ , is given by the expression

$$\begin{aligned}
 [\Sigma(\mathbf{p})]_{\alpha\beta,\gamma\delta} &= \frac{1}{2}w^2\frac{1}{N}\sum_{\mathbf{q}}\sum_{\substack{\mu\neq\alpha,\beta \\ \nu\neq\gamma,\delta}} [G_{\alpha\mu,\gamma\nu}(\mathbf{q})G_{\beta\mu,\delta\nu}(\mathbf{p}-\mathbf{q}) + G_{\alpha\mu,\delta\nu}(\mathbf{q})G_{\beta\mu,\gamma\nu}(\mathbf{p}-\mathbf{q}) \\
 &\quad + \{\mathbf{q} \leftrightarrow \mathbf{p}-\mathbf{q}\}]
 \end{aligned} \tag{25}$$

where the third and fourth terms inside the curly brackets, i.e. $\{\mathbf{q} \leftrightarrow \mathbf{p}-\mathbf{q}\}$, are the same as the first and second ones, but with \mathbf{q} and $\mathbf{p}-\mathbf{q}$ transposed.

5. Calculation of the correction to the order parameter function

In this section, the basic formulae for the calculation of the order parameter function $q(x)$ in two-loop order are presented. In what follows q_1 , x_1 , and $\Delta\bar{q}(r)$ — with $r = x/x_1$, and $\Delta\bar{q}(1) = 0$ exactly — are expressed by the parameters of the model in (1), namely τ (temperature), w (coupling constant), and Λ (momentum cutoff).

The equation between q_1 and τ is obtained by the first derivative of Eq. (3) evaluating at $x = x_1$:

$$\tau - wq_1 = -\frac{1}{2q_1}\left[\frac{d}{dr}(X_1 + X_2)\right]_{r=1} \tag{26a}$$

where Eq. (21a) and the definitions in (13) and (22) were used. The second derivative of (3) together with Eq. (21b) provides x_1 as:

$$x_1 = \frac{1}{2wq_1^2}\left[\frac{d^2}{dr^2}(X_1 + X_2)\right]_{r=1}. \tag{26b}$$

Using (21b) again, but now for generic r , and also (26b), one gets the correction to the leading, purely linear order parameter function:

$$\Delta\bar{q}(r) = \frac{1}{2wq_1^2x_1}\left\{\left[\frac{d^2}{dr^2}(X_1 + X_2)\right]_{r=1} \times r - \left[\frac{d^2}{dr^2}(X_1 + X_2)\right]\right\}. \tag{26c}$$

5.1. The calculation of x_1

5.1.1. Generic dimensions $6 < d < 8$

Applying the results of Section 4.2 and Appendix C, one can write for x_1 in Eq. (26b):

$$\begin{aligned}
 x_1 &= \frac{1}{2} \bar{\Lambda}^\epsilon \bar{g} \frac{d^2}{dr^2} \{F_0^{(l)} + x_1 [F_1^{(l)} + x_1^{-\epsilon/2} F_0^{(s)}]\} \Big|_{r=1} + \frac{1}{2wq_1^2} \left[\frac{d^2}{dr^2} X_2 \right]_{r=1} \\
 &= C_d \bar{\Lambda}^\epsilon \bar{g} + (K_1 \bar{\Lambda}^{-\epsilon} + K_2 + K_3 x_1^{-\epsilon/2}) x_1^2,
 \end{aligned} \tag{27}$$

with some d -dependent constants K_1 , K_2 , and K_3 . Since one can use the leading behavior $\bar{\Lambda}^\epsilon \bar{g} \sim x_1$ in the correction terms, these constants get contributions from both X_1 and X_2 . Rearranging this equation provides $\bar{\Lambda}^\epsilon \bar{g}$ in terms of x_1 and \bar{g} :

$$\begin{aligned}
 \bar{\Lambda}^\epsilon \bar{g} &= (C_d^{-1} - K_1 \bar{g} + \dots) x_1 + (-C_d^{-1} K_2 + \dots) x_1^2 + (-C_d^{-1} K_3 + \dots) x_1^{2-\epsilon/2} \\
 &\equiv f^{(1)}(\bar{g}) x_1 + f^{(2)}(\bar{g}) x_1^2 + f^{(na)}(\bar{g}) x_1^{2-\epsilon/2} \equiv \mathcal{F}(x_1, \bar{g}).
 \end{aligned} \tag{28}$$

The left-hand side is a measure of the distance from the critical temperature: $\bar{\Lambda}^\epsilon \bar{g} \sim \tau^{-\epsilon/2}$, note that $\epsilon < 0$, and the temperature dependence of x_1 is obtained by finding the root of the above equation. It can be easily seen that, when approaching the critical temperature, the ratio of the correction term of x_1 to the leading one is proportional to \bar{g} . Although this is the usual behavior above the upper critical dimension, the nonanalytical term, i.e. $f^{(na)}(\bar{g}) x_1^{2-\epsilon/2}$, is a peculiarity of the spin glass field theory.

5.1.2. At the upper critical dimension: $d = 6$

One can evaluate (27) at exactly $\epsilon = 0$ by using Eq. (18) for the X_1 part, furthermore Eqs. (C.2), (C.4), (C.5), (C.6), (C.7), and (C.8) from Appendix C for the X_2 part:

$$x_1 = 6\bar{g} - \frac{1}{3} x_1^2 [\ln \bar{\Lambda} + 2 \ln x_1 + O(1)] = 6\bar{g} - 12\bar{g}^2 [\ln \bar{\Lambda} + 2 \ln \bar{g} + O(1)] + \dots \tag{29}$$

where $C_{d=6} = 6$ has been used.² Evidently, the correction term blows up when approaching the critical temperature, demonstrating the well-known phenomenon that the simple perturbative method breaks down at the upper critical dimension. Nevertheless, one can make a comparison with the expanded form of the renormalization group result in Eq. (34) of Ref. [6]:

$$x_1 = 6 \left[\frac{\bar{g}}{1 - \bar{g} \ln(\tau \bar{g}^{-5/3} / \Lambda^2)} \right] + \dots = 6\bar{g} - 12\bar{g}^2 \ln \bar{\Lambda} - 10\bar{g}^2 \ln \bar{g} + \dots$$

where the equality, in leading order, of $\ln \bar{\Lambda} \equiv \ln(\Lambda / \sqrt{2wq_1})$ with $-\ln(2\tau / \Lambda^2) / 2$ has been used. The discrepancy in the $\bar{g}^2 \ln \bar{g}$ term arises from two factors: Firstly, from the fact that only the linear leading term of the scaling function $\hat{x}_1(x)$ was computed in [6] which is, however, sufficient for getting the $\bar{g}^2 \ln \tau$ contribution. Secondly, the $\bar{g}^2 \ln \bar{g}$ term in (29) comes from the small mass regime, and it is not expected to be controlled by renormalization around the critical theory. Considering that in Ref. [6] x_1 was computed by an expansion around the replica symmetric field theory — in contrast to the present fully RSB calculation — the agreement found in the $\bar{g}^2 \ln \tau$ term is rather reassuring.

5.1.3. Below the upper critical dimension: $d = 6 - \epsilon$

One can revive perturbation theory below six dimensions by the method of the ϵ -expansion: the d -dimensional integrals are expanded in ϵ , and the coupling constant \bar{g} must be specially

² The $x_1^2 \ln x_1$ contributions from $(d^2/dr^2 X_1)_{r=1}$ and $(d^2/dr^2 X_2)_{r=1}$ with the Σ_R^{xx} self-energy insertion cancel each other — a surprising effect whose origin is not understood. The $-2/3x_1^2 \ln x_1$ above comes from the Σ_L^{xx} , $\Sigma_x^{x_1 x_1}$, and $\Sigma_1^{x_1 x_1}$ self-energy components.

chosen for having correctly exponentiating logarithms. In this second order calculation, one can stay in six dimensions when evaluating the correction terms. As a result, Eq. (29) is only slightly modified:

$$\begin{aligned}
 x_1 &= 6\bar{g}(1 + \epsilon \ln \bar{\Lambda}) - \frac{1}{3}x_1^2[\ln \bar{\Lambda} + 2 \ln x_1 + O(1)] \\
 &= 6\bar{g} + 6\bar{g}(\epsilon - 2\bar{g}) \ln \bar{\Lambda} - \frac{1}{3}x_1^2[2 \ln x_1 + O(1)]
 \end{aligned}
 \tag{30}$$

where $C_d = 6 + O(\epsilon^2)$ was used. The correction term becomes infinite at criticality, except for $2\bar{g} = \epsilon + O(\epsilon^2)$ which is just the fixed point condition in the first order of the perturbative renormalization group, see for instance [6].³ In this way, the equation that determines x_1 becomes independent of the temperature, and x_1 itself is nonzero and universal below six dimensions:

$$3\epsilon = x_1 + \frac{1}{3}x_1^2[2 \ln x_1 + O(1)].
 \tag{31}$$

In a generic dimension $d < 6$, there should exist a function $\hat{\mathcal{F}}(x_1, \hat{g})$ such that x_1 is determined from

$$\hat{\mathcal{F}}(x_1, \hat{g}) = 0
 \tag{32}$$

where \hat{g} is the special coupling as explained in footnote 3.

5.2. The Edwards–Anderson order parameter

5.2.1. q_1 in dimensions $6 < d < 8$

After evaluating Eq. (26a), one gets:

$$\begin{aligned}
 \tau - wq_1 &= -\frac{1}{2}(wq_1)\bar{\Lambda}^\epsilon \bar{g} \left\{ -4 \int_0^{\bar{\Lambda}} dp p^{d-1} G_R G_L^2 + C_d + x_1 \frac{d}{dr} [F_1^{(l)} + x_1^{-\epsilon/2} F_0^{(s)}]_{r=1} \right\} \\
 &\quad - \frac{1}{2q_1} \left[\frac{d}{dr} X_2 \right]_{r=1} \\
 &= (wq_1) \left\{ \left(-\frac{2}{\epsilon} + \frac{1-2\epsilon}{3\epsilon} C_d \bar{\Lambda}^\epsilon \right) \bar{g} + (K'_1 \bar{\Lambda}^{-2\epsilon} + K'_2 \bar{\Lambda}^{-\epsilon} + K'_3 + K'_4 x_1^{-\epsilon/2}) x_1^2 \right\} \\
 &= (wq_1) \left\{ \left(-\frac{2}{\epsilon} + \frac{1-2\epsilon}{3\epsilon} C_d \bar{\Lambda}^\epsilon \right) \bar{g} + (K'_1 + K'_2 \bar{\Lambda}^\epsilon + K'_3 \bar{\Lambda}^{2\epsilon}) C_d^2 \bar{g}^2 + K'_4 x_1^{2-\epsilon/2} \right\},
 \end{aligned}
 \tag{33}$$

with some d -dependent constants K'_1, \dots, K'_4 [see the remark below Eq. (27) which is appropriate here too]. The terms first and second order in the dimensionless coupling constant \bar{g} in

³ More importantly, we will see in the next subsection that the very same condition ensures that the $\ln \bar{\Lambda}$'s correctly exponentiate, at least in the order we are studying here, when the τ versus q_1 relation is computed perturbatively below six dimensions. One may expect that a choice $\bar{g} = \hat{g}$ will guarantee in *any order*: (i) the vanishing of the $\ln \bar{\Lambda}$'s in the series for x_1 , and (ii) the correct exponentiation of them in the τ versus q_1 series. Nevertheless, \hat{g} is not needed to be identical with the fixed point, except in first order in ϵ , since \bar{g} is a *bare* dimensionless coupling.

the last line are quite usual in the perturbative expansion of an order parameter above the upper critical dimension, whereas the nonanalytic last term has its origin in the far infrared regime, and is specific to the spin glass field theory with the two distinct bands of the mass spectrum.

5.2.2. q_1 in six dimensions

For finding q_1 perturbatively in $d = 6$, one can use (18):

$$\frac{d}{dr} [F_1^{(l)} + x_1^{-\epsilon/2} F_0^{(s)}]_{r=1} = 4 \ln \bar{\Lambda} + \frac{74}{15} \ln x_1 + O(1),$$

and collect terms from (C.2), (C.4), (C.5), (C.6), (C.7), and (C.8):

$$\frac{1}{q_1} \left[\frac{d}{dr} X_2 \right]_{r=1} = (wq_1)x_1^2 \left[-\frac{2}{3} \ln x_1 + O(1) \right].$$

[It is remarkable that the different terms for $\ln^2 \bar{\Lambda}$ and $\ln \bar{\Lambda}$ in $(\frac{d}{dr} X_2)_{r=1}$ all cancel each other, and the $-2/3 \ln x_1$ comes from the Σ_R^{xx} component, all the other contributions are zero.] Putting these second order terms into (33), and evaluating the first order one at $d = 6$, one gets:

$$\tau - wq_1 = (wq_1) \left\{ (2 \ln \bar{\Lambda} - 4) \bar{g} - 6 \left[2 \ln \bar{\Lambda} + \frac{7}{15} \ln \bar{g} + O(1) \right] \bar{g}^2 \right\}$$

where $\ln x_1$ was substituted by $\ln(6\bar{g})$. This equation gives τ versus q_1 for a fixed temperature below T_c . For getting the behavior of q_1 for τ approaching zero for a given system (i.e. for some given \bar{g}), one must resum an infinite series: this is just done by the renormalization group, as in Ref. [6].

5.2.3. ϵ -expansion for q_1 , $d < 6$

The first term in the second row of Eq. (33) must now be expanded up to $O(\epsilon)$, while the second and third ones can be evaluated at six dimensions. We get for τ :

$$\begin{aligned} \tau &= (wq_1) \left\{ 1 + 2(\ln \bar{\Lambda} - 2) \bar{g} + \left(\ln^2 \bar{\Lambda} - 4 \ln \bar{\Lambda} + \frac{\pi^2}{12} \right) \epsilon \bar{g} \right. \\ &\quad \left. - 6 \left[2 \ln \bar{\Lambda} + \frac{7}{15} \ln \bar{g} + O(1) \right] \bar{g}^2 \right\} \\ &= (wq_1) \left\{ (1 - 4\bar{g}) + (2\bar{g} - 4\epsilon\bar{g} - 12\bar{g}^2) \ln \bar{\Lambda} + \epsilon \bar{g} \ln^2 \bar{\Lambda} \right. \\ &\quad \left. + \left[-\frac{14}{5} \bar{g}^2 \ln \bar{g} + \frac{\pi^2}{12} \epsilon \bar{g} + O(\bar{g}^2) \right] \right\}. \end{aligned} \tag{34}$$

The first three terms in the curly brackets can be considered as part of the expansion of $(1 - 4\bar{g} + \dots) \bar{\Lambda}^\kappa$, provided that $(1 - 4\bar{g})\kappa = 2\bar{g} - 4\epsilon\bar{g} - 12\bar{g}^2$ and $\frac{1}{2}\kappa^2 = \epsilon\bar{g}$, yielding the exponentiation condition $\bar{g} = \hat{g} = \frac{1}{2}\epsilon$ — which coincides with the requirement of vanishing $\ln \bar{\Lambda}$ contribution to x_1 found in the previous subsection, see footnote 3 —, and $\kappa = 2\hat{g} - 12\hat{g}^2 + O(\hat{g}^3)$. Identifying the critical exponent β as $\beta^{-1} = 1 - \frac{1}{2}\kappa$, it is obtained:

$$\beta = 1 + \hat{g} - 5\hat{g}^2 + \dots = 1 + \frac{1}{2}\epsilon + O(\epsilon^2). \tag{35}$$

Unfortunately only the leading behavior of \hat{g} is available at the moment, preventing us from computing β up to $O(\epsilon^2)$, and compare it with known results from renormalization in the symmetric

(high-temperature) theory [14,15]. (See also Ref. [8] where this first order result for β has been presented from calculation in the glassy phase.)

5.3. The correction to the order parameter function: $\Delta\bar{q}(r)$

Eq. (26c) shows that $\Delta\bar{q}(1) = 0$, as it must be. Furthermore, one can conclude from this formula that terms linear ($\sim r$) and cubic ($\sim r^3$) in X_1 and X_2 give no contributions to $\Delta\bar{q}(r)$. Using Eqs. (13), (14), (16), and (17), it then immediately follows:

$$\Delta\bar{q}(r) = \frac{1}{2}\bar{A}^\epsilon\bar{g} \left\{ \left[\frac{d^2}{dr^2}(F_1^{(l)} + x_1^{-\epsilon/2}F_0^{(s)}) \right]_{r=1} \times r - \left[\frac{d^2}{dr^2}(F_1^{(l)} + x_1^{-\epsilon/2}F_0^{(s)}) \right] \right\} + \frac{1}{2wq_1^2x_1} \left\{ \left[\frac{d^2}{dr^2}X_2 \right]_{r=1} \times r - \left[\frac{d^2}{dr^2}X_2 \right] \right\}. \tag{36}$$

- As far as a generic dimension $6 < d < 8$ is concerned, one can compute the first part of (36) by use of Eq. (17):

$$\Delta\bar{q}(r) = x_1 C_d^{-1} \left[-57(r - r^3) \int_0^\infty dp p^{d-1} G_R^3 G_L^2 + 16(r - r^4) \int_0^\infty dp p^{d-1} G_R^3 G_L^3 \right] + O(x_1^{1-\frac{\epsilon}{2}}).$$

Only terms proportional to r^5 give contributions to the second part of (36) — these come from Σ_R^{xx} , Σ_L^{xx} , $\Sigma_x^{x_1x_1}$, $\Sigma_1^{xx_1}$, and $\delta\Sigma^{xx_1}$ — yielding

$$\frac{1}{2wq_1^2x_1} \left\{ \left[\frac{d^2}{dr^2}X_2 \right]_{r=1} \times r - \left[\frac{d^2}{dr^2}X_2 \right] \right\} \sim x_1(r - r^3) \times [\text{convergent 2-loop integrals}],$$

and also a complicated nonanalytical contribution of order $x_1^{1-\frac{\epsilon}{2}}$, coming from integrals in the far infrared region, which is negligible for $d > 6$, but becomes more and more important when approaching $d = 6$.

One can conclude from this 2-loop calculation that the order parameter function takes the form

$$q(x)/q_1 = (1 + a_d x_1 + \dots)r + x_1(c_d r^3 + d_d r^4) + \Delta\bar{q}^{(na)}(r) + \dots, \quad r = x/x_1, \quad 6 < d < 8$$

where the nonanalytical contribution is subleading in this dimensional regime:

$$\Delta\bar{q}^{(na)}(r) \sim x_1^{1-\frac{\epsilon}{2}}.$$

All the temperature dependence of $q(x)$ is absorbed into q_1 and x_1 which are, as it follows from our scheme, the exact Edwards–Anderson order parameter and breakpoint of $q(x)$. Furthermore, although $q(x)/q_1$ should, in principle, depend on both x_1 and \bar{g} , it proved to be, at least up to the order considered, \bar{g} -independent.

The emergence of the $x_1 \times d_d r^4$ term, coming solely from the one-loop graph X_1 , may seem somewhat surprising, although a similar x^4 contribution has been found in the mean

field order parameter function [16]. (But see Section 7 for confronting the results here for $6 < d < 8$ with their mean field counterparts, i.e. $d = \infty$.)

- In exactly six dimensions, (36) can be evaluated by use of Eqs. (18), (C.2), (C.4), (C.5), and (C.6); note that only terms proportional to r^5 and the sole r^6 contribution from X_1 give nonvanishing result:

$$\Delta\bar{q}(r) = -\left[\frac{1}{4}(r - r^3) + \frac{4}{3}(r - r^4)\right]x_1 \ln x_1 + O(x_1), \quad d = 6.$$

6. The study of the momentum-dependent mass

By Dyson’s equation, the inverse of the exact propagator is identical with the mass operator $\Gamma(\mathbf{p})$, and stability of the RSB phase demands that the eigenvalues of $\Gamma(\mathbf{p} = \mathbf{0})$ be all nonnegative. Eqs. (4) and (7) give the elements of the mass operator as:

$$\begin{aligned} \Gamma_{\alpha\beta,\alpha\beta} &= p^2 - 2wq_1 + x_1wq_1 + \delta M - \Sigma_{\alpha\beta,\alpha\beta}(\mathbf{p}), \\ \Gamma_{\alpha\gamma,\beta\gamma} &= -wq_{\alpha\beta} - \Sigma_{\alpha\gamma,\beta\gamma}(\mathbf{p}), \\ \Gamma_{\alpha\beta,\gamma\delta} &= -\Sigma_{\alpha\beta,\gamma\delta}(\mathbf{p}), \end{aligned} \tag{37}$$

and everything above is understood to be exact quantity (the self-energy, for instance, although the same notation is used as for its first order part throughout the paper). Instead of a full analysis, our study will be confined to the highest longitudinal eigenvalue and to the family of the replicon ones.

6.1. The longitudinal mass

Applying the results from Ref. [17], an eigenvector $f_{\alpha\beta}$ of the longitudinal subspace has the same ultrametric structure as the order parameter $q_{\alpha\beta}$, and we can use a similar parametrization for it, i.e.

$$f_{\alpha\beta} = f_{\alpha\cap\beta} = f(x) = f_1\bar{f}(r) = f_1[r + \Delta\bar{f}(r)], \quad \text{where } r = x/x_1, \text{ and } \Delta\bar{f}(1) = 0.$$

The eigenvalue equation can now be written as

$$\begin{aligned} \frac{1}{2} \sum_{\gamma \neq \delta} \Gamma_{\alpha\beta,\gamma\delta} f_{\gamma\delta} &= (p^2 - 2wq_1 + x_1wq_1 + \delta M)\bar{f}(r) \\ &+ 2wq_1[\bar{q}(r) + \bar{f}(r) - x_1\{\bar{q}(r), \bar{f}(r)\}] \\ &- \Sigma_R^{xx}(\mathbf{p})\bar{f}(r) + 2\delta\Sigma^{xx_1}(\mathbf{p}) = \lambda\bar{f}(r). \end{aligned}$$

The SBA has been used in the terms with the one-loop self-energy components, which is correct at the present level of approximation. It is easy to check that $\lambda = p^2 + 2wq_1$ and $\bar{f}(r) = r$ are the zeroth order solutions. Due to the x_1 in front of the bilinear form, see (11) for its definition, it is sufficient to replace \bar{q} and \bar{f} by r : $\{\bar{q}(r), \bar{f}(r)\} \approx \{r, r\} = r - r^3/3$. Setting $r = 1$ and using Eq. (A.2) yield the first order result for the longitudinal momentum-dependent mass:

$$\lambda \equiv \Gamma_{\text{long}}(\mathbf{p}) = p^2 + 2wq_1 - \frac{4}{3}x_1wq_1 - [\Sigma_R^{x_1x_1}(\mathbf{p}) - \Sigma_R^{x_1x_1}(\mathbf{p} = \mathbf{0})] + 2\delta\Sigma^{x_1x_1}(\mathbf{p}). \tag{38}$$

By the help of Eqs. (B.8) and (B.9), and the row for $\delta\Sigma^{xx_1}$ in Table 1, the (zero-momentum) longitudinal mass above six dimensions is as follows:

Table 1

The most relevant properties of the self-energy components. The functions f , $\sigma^{(a)}$, and $f^{(l)}$ — their argument being the dimensionless momentum squared, i.e. $p^2/(2wq_1) \rightarrow p^2$ — and the exponents a , b are defined in Eqs. (B.8) and (B.9). While $f(p^2)$ is exact, only the leading $1/\epsilon$ term for $\sigma^{(a)}$ and $f^{(l)}$ is shown ($\epsilon = 6 - d$). C_d is the notation for $6\Gamma(d/2 - 2)\Gamma(4 - d/2)$, whereas $r \equiv x/x_1$ throughout the paper.

	$f(p^2)$	$\sigma^{(a)}(p^2)$	$f^{(l)}(p^2)$	a	b
Σ_R^{xx}	$4C_d^{-1} \left(1 + \frac{4-d}{d} p^2\right) \frac{1}{\epsilon}$	$\left(\frac{2}{9} p^2 - \frac{2}{3}\right) \frac{1}{\epsilon} + O(1)$	$O(1)$	2	1
Σ_L^{xx}	$4C_d^{-1} \frac{4-d}{d} p^2 \frac{1}{\epsilon}$	$\left[\frac{2}{9} p^2 - \frac{2}{3}(1-r^2)^2 \frac{1}{p^2+1}\right] \frac{1}{\epsilon} + O(1)$	$\frac{2}{3}(1-r^2)^2 \frac{1}{p^2+1} \frac{1}{\epsilon} + O(1)$	0	0
$\Sigma_x^{x_1 x_1}$	0	$-\frac{2}{3}(1-r^2)^2 \frac{1}{p^2} \frac{1}{\epsilon} + O(1)$	$\frac{2}{3}(1-r^2)^2 \frac{1}{p^2} \frac{1}{\epsilon} + O(1)$	0	1
$\Sigma_1^{xx_1}$	$2C_d^{-1} r \frac{1}{\epsilon}$	$-\frac{1}{3} r \left[1 - (1-r^2)^2 \frac{1}{p^2(p^2+1)}\right] \frac{1}{\epsilon} + O(1)$	$-\frac{1}{3} r (1-r^2)^2 \frac{1}{p^2(p^2+1)} \frac{1}{\epsilon} + O(1)$	0	1
$\delta \Sigma^{xx_1}$	$2C_d^{-1} r \frac{1}{\epsilon}$	$-\frac{1}{3} r \frac{1}{\epsilon} + O(1)$	$O(1)$	1	1
$\Sigma_L^{x_1 x_1}$	$4C_d^{-1} \frac{4-d}{d} p^2 \frac{1}{\epsilon}$	$\frac{2}{9} p^2 \frac{1}{\epsilon} + O(1)$	$O(1)$	2	1
$\Sigma_{LA}^{x_1 x_1}$	$4C_d^{-1} \frac{1}{\epsilon}$	$-\frac{2}{3} \frac{1}{\epsilon} + O(1)$	$O(1)$	2	1

$$\Gamma_{\text{long}}(\mathbf{p} = \mathbf{0}) = 2wq_1 \left[1 + \frac{2}{\epsilon} \bar{g} + O(x_1) + O(x_1^{1-\epsilon/2}) \right], \quad 6 < d < 8. \quad (39)$$

Considering that the longitudinal momentum-dependent mass is the inverse of the longitudinal exact propagator, one can expect that its behavior below six dimensions is governed by the critical fixed point, and it has the following scaling form:

$$\Gamma_{\text{long}}(\mathbf{p}) = p^2(p/\Lambda)^{-\eta} \bar{g} \left[(p/\Lambda)^2 \bar{\Lambda}^{4\nu/\beta} \right], \quad d < 6. \quad (40)$$

One can check this scaling by evaluating the self-energy components in (38) at exactly six dimensions:

$$\begin{aligned} \Sigma_R^{xx}(\mathbf{p}) - \Sigma_R^{xx}(\mathbf{p} = \mathbf{0}) = & \bar{g} w q_1 \left\{ \frac{1}{9} (p^2/2wq_1) [12 \ln(\Lambda/p) + 11] - [4 \ln(\Lambda/p) + 1] \right. \\ & + 4(\ln \bar{\Lambda} - 2) + 6(1 - r^2) \\ & + 2(1 + 2r^2)(p^2/2wq_1)^{-1} \ln(p^2/2wq_1) \\ & \left. + (3 + 4r^2)(p^2/2wq_1)^{-1} + O[(p^2/2wq_1)^{-2}] \right\}, \end{aligned}$$

and

$$\begin{aligned} \delta \Sigma^{x_1 x_1}(\mathbf{p}) = & \bar{g} w q_1 \left\{ -\frac{1}{2} [4 \ln(\Lambda/p) + 1] - 5(p^2/2wq_1)^{-1} \ln(p^2/2wq_1) \right. \\ & \left. - \frac{1}{2} (p^2/2wq_1)^{-1} + O[(p^2/2wq_1)^{-2}] \right\}. \end{aligned}$$

Inserting these expressions into (38) and using $x_1 = 6\bar{g}$, one gets:

$$\eta = -\frac{2}{3}\bar{g}, \quad \frac{4\nu}{\beta} = 2 + \frac{4}{3}\bar{g}, \quad \text{and}$$

$$\mathcal{G}(u) = (1 + u^{-1}) + \left[-\frac{11}{18} - \frac{1}{3}u^{-1} \ln u - 4u^{-2}(2 \ln u + 1) + \dots \right] \bar{g},$$

in full agreement with the ϵ -expansion results from calculations in the symmetric (high temperature) phase in Ref. [15], whenever \bar{g} is substituted by $\hat{g} = \epsilon/2$, namely

$$\eta = -\frac{1}{3}\epsilon, \quad \nu = \frac{1}{2} \left(1 + \frac{5}{6}\epsilon \right), \quad \text{and} \quad \beta = 1 + \frac{1}{2}\epsilon.$$

The zero-momentum limit of Γ_{long} is the inverse longitudinal susceptibility. By use of Eq. (40), one gets its behavior:

$$\Gamma_{\text{long}}(\mathbf{p} = \mathbf{0}) \sim \bar{\Lambda}^{-\frac{2\nu(\eta-2)}{\beta}} \sim \bar{\Lambda}^{-\frac{-2\gamma}{\beta}} \sim \tau^\gamma, \quad d < 6.$$

By means of Eqs. (B.5) and (A.1), it follows that

$$\delta \Sigma^{x_1 x_1}(\mathbf{0}) = 2C_d^{-1} x_1(wq_1) \int_0^{\bar{\Lambda}} dp p^{d-1} (-G_R^2 G_L - 8G_R^2 G_L^2 + 8G_R^2 G_L^3)$$

$$\stackrel{d=6}{=} -\frac{1}{3} x_1(wq_1) (\ln \bar{\Lambda} + 2)$$

where the final result was obtained by neglecting irrelevant, i.e. $\sim \bar{\Lambda}^{-2}$, terms. The zero-momentum limit of (38) can now be written as:

$$2wq_1 - \frac{8}{3}x_1(wq_1) - \frac{2}{3}x_1(wq_1) \ln \bar{\Lambda} = 2wq_1(1 - 8\bar{g})\bar{\Lambda}^{-2\bar{g}}, \quad x_1 = 6\bar{g} + \dots$$

One can then conclude that $\gamma/\beta = 1 + \bar{g}$ and, using (35), $\gamma = 1 + 2\bar{g}$. This yields — again in full agreement with Ref. [15] — $\gamma = 1 + \epsilon$, after the special condition $\bar{g} = \hat{g}$ for the coupling constant has been applied.

6.2. The replicon band of $\Gamma(\mathbf{p})$

It was shown in Refs. [8,17] that the replicon eigenvalues of any ultrametric matrix can be easily computed by direct substitution of the matrix elements into an expression such as Eq. (41) in [17]. Inserting the components of $\Gamma(\mathbf{p})$ in Eq. (37) into this formula, and keeping terms up to first order in x_1 , a surprisingly simple result is obtained:

$$\Gamma_{\text{repl}}(x; u, v) = p^2 + x_1(wq_1) [(r_1^2 + r_2^2)/2 - r^2] - [\Sigma_R^{xx}(\mathbf{p}) - \Sigma_R^{xx}(\mathbf{p} = \mathbf{0})]$$

where $0 \leq r = x/x_1 \leq r_1 = u/x_1$, $r_2 = v/x_1 \leq 1$, and Eq. (A.2) was applied. The middle term is just the zero-momentum replicon mass; marginal stability is clearly demonstrated. The $u = v = x$ mode is known exactly [18] being a zero-(Goldstone)mode, and we can see this here perturbatively:

$$\Gamma_{\text{repl}}(x; x, x) = 0 + O(x_1^2), \quad 0 \leq x \leq x_1, \quad \mathbf{p} = \mathbf{0}.$$

One can suspect that this marginality persists, and is satisfied order by order in the perturbation expansion.

7. Discussion of the results, and some conclusions

The glassy phase of the replica field theory representing the Ising spin glass has a special dimensionless parameter, x_1 , not present in ordinary field theories, which is a characteristic of the RSB low-temperature phase. Just below T_c and in systems above the upper critical dimension (more precisely for $6 < d < 8$) x_1 is related to the other two dimensionless quantities, namely $\bar{\Lambda} = \Lambda/(2wq_1)^{\frac{1}{2}}$ (which diverges at criticality) and $\bar{g} = w^2 K_d/\Lambda^\epsilon$ (the dimensionless coupling constant), by an equation like (27) or (28). It has been shown in Ref. [6] that x_1 becomes nonzero at criticality, i.e. it is independent of $\bar{\Lambda}$, and universal below the upper critical dimension. The d -dependence of this universal value is calculated in this paper up to second order in ϵ , see (31). The equation for generic d can be written as in Eq. (32), where the special coupling constant \bar{g} insures proper exponentiation of temperature singularities, and it is related, but not equivalent, to the fixed point of the Wilson-type perturbative renormalization group.

In the classical perturbative regime above the upper critical temperature the behavior of $\tau/(wq_1)$ and $\Gamma_{\text{long}}(\mathbf{p} = \mathbf{0})/(2wq_1)$ are displayed in Eqs. (33) and (39). They both are the sum of a regular and an anomalous term. The regular ones have the following common structure: $\sum_{i,j} c_{ij} \bar{g}^i x_1^j$, and the c_{ij} coefficients for $i + j = L$ can be calculated by an L -loop calculation (higher loop terms do not change the result), notwithstanding that the free propagator itself is an infinite series in x_1 . On the other hand, the anomalous term is nonanalytic in x_1 : it is proportional to $x_1^{2-\epsilon/2}$ and $x_1^{1-\epsilon/2}$ in the two cases. It is argued in the following that this anomalous contribution (which comes always from far infrared integration) is nonperturbative in the sense that higher loop graphs yield similar anomalous terms. Let us look at, as an example, the following $k + 1$ -loop contribution to the equation of state in (3), see also (6):

$$\begin{aligned} X^{(k)} &\equiv w \frac{1}{N} \sum_{\mathbf{p}} \sum_{\gamma \neq \alpha, \beta} \{G[(\Sigma - M_1)G]^k\}_{\alpha\gamma, \beta\gamma} \\ &= -2w \frac{1}{N} \sum_{\mathbf{p}} [G_R^{xx} G_1^{xx_1} + (2G_R^{xx} - G_L^{xx}) \delta G^{xx_1}] \{[(\Sigma - M_1)G]^k G^{-1}\}_R^{xx} \end{aligned}$$

where the SBA was used, and only the replicon contribution is considered here, see also (24). Since $\{ \dots \}_R^{xx}$ is an exact eigenvalue of *any generic* ultrametric matrix, we can write:

$$\{[(\Sigma - M_1)G]^k G^{-1}\}_R^{xx} = [(\Sigma - M_1)_R^{xx}]^k \times [G_R^{xx}]^{k-1}.$$

By the help of Eqs. (7), (A.2), (B.8), (B.9), and Table 1, one gets for momenta in the far infrared, i.e. $\frac{p^2}{x_1(2wq_1)} = u$ with $u = O(1)$:

$$\begin{aligned} (\Sigma - M_1)_R^{xx} &= x_1(wq_1) \{r^2 + [f(ux_1) - f(0)]\bar{\Lambda}^{-\epsilon} + [\sigma^{(a)}(ux_1) - \sigma^{(a)}(0)] \\ &\quad + x_1^{1-\epsilon/2} [f^{(s)}(u) - f(0)]\} \\ &\approx x_1(wq_1)r^2, \end{aligned}$$

the approximation meaning leading order in x_1 . For the replicon free propagator one has exactly:

$$G_R^{xx} = \frac{1}{p^2 + \lambda_0(x; x_1, x_1)} = \frac{1}{p^2 + x_1 w q_1}$$

where Eq. (8) has been used for getting the last equality. Finally, using the classification of far infrared propagators at the end of Appendix A and Eq. (A.4), $X^{(k)}$ has the following leading contribution in the small mass regime:

$$\begin{aligned}
 X^{(k)} &= -2w \frac{1}{N} \sum_{\mathbf{p}} \frac{1}{x_1 p^4} G^{(s)} \left[\frac{p^2}{x_1 (2wq_1)} \right] (x_1 wq_1)^k r^{2k} \frac{1}{(p^2 + x_1 wq_1)^{k-1}} \\
 &\sim (wq_1^2) x_1^{1-\epsilon/2} (\bar{g} \bar{\Lambda}^\epsilon) r^{2k} \sim (wq_1^2) x_1^{2-\epsilon/2} r^{2k}
 \end{aligned}$$

where the last formula was obtained by use of (19b). The order of this formula, i.e. the power of x_1 , is independent of the number of loops $k + 1$, and $\tau/(wq_1)$ gets a nonanalytic contribution which is proportional to $x_1^{2-\epsilon/2}$ in any order of the loop-expansion.

What has been learnt from the study of the classical perturbative regime is extensible to the case of $d < 6$. $\tau/(wq_1)$ and $\Gamma_{\text{long}}(\mathbf{p} = \mathbf{0})$ can be separated into a regular part (coming from integration in the near infrared), and an anomalous one originating from the small mass (far infrared) sector. The regular part is under the control of the critical fixed point, and usual critical exponents can be computed, after the separation has been done, as in Sections 5.2.3 and 6.1. The dangerous infrared behavior is cut off by the small mass, and it is isolated into the anomalous part; see as an example the $\bar{g}^2 \ln \bar{g}$ term in Eq. (34).

$\Delta\bar{q}(r) = q(x)/q_1 - r$, $r = x/x_1$, has been assumed in Ref. [6] to be proportional to x_1^2 , just as in mean field theory. It turns out that this is true only in high spatial dimensions, namely for $d > 10$. A preliminary study suggests that $\Delta\bar{q}(r) \sim x_1^{d/2-3}$ in the dimensional range of $8 < d < 10$. In this paper, we have computed $\Delta\bar{q}(r)$ in leading order, and it proved to be of order x_1 in the whole range of $6 < d < 8$. Beside this regular term, far infrared integration provides a nonanalytic (anomalous) contribution of the form $\Delta\bar{q}(r) \sim x_1^{1-\epsilon/2}$, which is again expected to emerge in higher order too. Another remarkable finding of the present calculation is that $\Delta\bar{q}(r)$ depends only on x_1 , and not on \bar{g} (note that $d > 6$): this may be a generic property of $q(x)$ for $d < 8$.

Finally, we obtained an important result for the family of the (zero-momentum) replicon mass, namely

$$\begin{aligned}
 \Gamma_{\text{repl}}(x; u, v) &= x_1 (wq_1) [(r_1^2 + r_2^2)/2 - r^2], \\
 0 \leq r = x/x_1 \leq r_1 = u/x_1, \quad r_2 = v/x_1 \leq 1.
 \end{aligned}$$

This formula is in complete agreement with that of the truncated model of mean field theory [8,19], any effect of the short range interaction and the geometry of the hypercubic lattice is embedded in x_1 and q_1 . Stability of the ultrametric RSB phase below six dimensions is thus demonstrated along the same lines as in mean field theory.

As an important task in the near future, one should reanalyze the small momentum behavior of $G_{11}^{xx}(\mathbf{p})$ (this is the object studied numerically in three dimensions in Refs. [20–22]), and finding out how it changes when crossing the upper critical dimension. Now this seems to be feasible by the knowledge collected for the first order self-energy in Appendix B.

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Appendix A. Some results for the free propagators in the near and far infrared regimes

Equations like (24) and (B.1) to (B.7) are complicated but manageable in the large mass regime where everything can be expressed in terms of the two propagators introduced in (15).

Here the relevant propagator components⁴ are listed in dimensionless form, and, for easing the notation, we keep the old notations for the dimensionless quantities: i.e. $(2wq_1)G \rightarrow G$ and $p^2/(2wq_1) \rightarrow p^2$. All the results below are taken from Ref. [8], see also (23):

$$\begin{aligned}
 G_R^{xx} &= G_R, & G_L^{xx} &= G_L[1 - (1 - r^2)G_R^2], \\
 G_{LA}^{xx} &= G_L^2 \left[1 + \frac{3}{2}(1 - r^2)G_R - \frac{1}{2}(1 - r^2)^2 G_R^3 \right], \\
 G_1^{xx_1} &= \frac{1}{2}rG_R G_L [(1 + 2G_L) + (1 - r^2)G_R G_L], \\
 \delta G^{xx_1} &= \frac{1}{2}rG_R G_L, & G_x^{x_1x_1} &= r^2 G_R G_L^2.
 \end{aligned}
 \tag{A.1}$$

As an application of these formulae, the leading contribution of the replicon self-energy (B.1) at zero momentum is easily derived as

$$\Sigma_R^{xx}(\mathbf{p} = \mathbf{0}) = \delta M + x_1 w q_1 r^2
 \tag{A.2}$$

where the integrals obtained were substituted by the results in Eqs. (19b) and (20).

We can also use the leading large mass propagators in (24), and after inserting the expressions from (A.1), we get the following result valid in the near-infrared ($p^2 \sim 2wq_1$) regime:

$$\begin{aligned}
 &(2wq_1)^2 (G \Sigma G - \delta M G^2)_1^{xx_1} \\
 &= \frac{1}{2}r \left[(2G_R^3 + G_R^2 G_L^2) - 2r^2 G_R^3 G_L^2 \right] (\Sigma_R^{xx} - \delta M) - \frac{1}{2}r G_R^2 G_L (\Sigma_L^{xx} - \delta M) \\
 &\quad - \frac{1}{2}r G_R G_L^2 \Sigma_x^{x_1x_1} + G_R G_L \Sigma_1^{xx_1} \\
 &\quad + \left[(2G_R G_L^2 + G_R^2 G_L^2) - 3r^2 G_R^2 G_L^2 \right] \delta \Sigma^{xx_1} \\
 &\quad + \frac{1}{2}r \left[(G_R G_L^2 + 3G_R^2 G_L^2 - 2G_R^2 G_L^3) - 2r^2 G_R^2 G_L^3 \right] (\Sigma_L^{x_1x_1} - \delta M) \\
 &\quad + \frac{1}{2}r G_R G_L^2 \Sigma_{LA}^{x_1x_1}.
 \end{aligned}
 \tag{A.3}$$

The free propagator components in the far infrared region (small mass regime) have the following leading term (restoring now the dimensional dependence of p^2 again):

$$G \sim \frac{1}{x_1^k p^2} G^{(s)} \left[\frac{p^2}{x_1(2wq_1)} \right],
 \tag{A.4}$$

and using the matching condition (see Section 3) between the near and far infrared regimes, one can infer the k exponent from the leading infrared power of the large mass form in (A.1):

$$G \sim \frac{1}{p^{2(1+k)}}.$$

The following classes are found:

- $k = 2$: G_{LA}^{xx} , $x < x_1$. This is the most infrared divergent propagator of all.
- $k = 1$: G_L^{xx} and $G_1^{xx_1}$, $x < x_1$.

⁴ More precisely, their leading terms.

- $k = 0$: G_R^{xx} , $G_1^{x_1x_1}$, δG^{xx_1} , and $G_x^{x_1x_1}$, $x \leq x_1$.
- $k = -1$: $G_L^{x_1x_1}$ and $G_{LA}^{x_1x_1}$.

Appendix B. The self-energy components appearing in Eq. (24)

A generic component of the one-loop self-energy matrix is shown in (25). In the present calculation, we can use the SBA when computing the self-energy components occurring in (24). These components are linear combinations according to the rules in Eq. (23), now applied for the self-energy matrix. For easing the notation, the momentum arguments are not displayed: as a general rule, the first G is always at momentum \mathbf{q} , whereas the second one in a product is at $\mathbf{p} - \mathbf{q}$. The interchange $\{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}$ means, just as in (25), the same terms but with interchanging \mathbf{q} and $\mathbf{p} - \mathbf{q}$. After some replica algebra one obtains:

$$\Sigma_R^{xx}(\mathbf{p}) = w^2 \frac{1}{N} \sum_{\mathbf{q}} [G_R^{xx} (G_x^{x_1x_1} - 2G_L^{x_1x_1} - G_{LA}^{x_1x_1}) + G_L^{xx} G_L^{x_1x_1} + 4\delta G^{xx_1} \delta G^{xx_1} + \{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}], \tag{B.1}$$

$$\Sigma_L^{xx}(\mathbf{p}) = w^2 \frac{1}{N} \sum_{\mathbf{q}} [G_R^{xx} (2G_x^{x_1x_1} + 3G_R^{x_1x_1} - 3G_L^{x_1x_1} - 2G_{LA}^{x_1x_1}) + G_L^{xx} (G_x^{x_1x_1} - G_L^{x_1x_1} - G_{LA}^{x_1x_1}) - 2G_{LA}^{xx} G_L^{x_1x_1} - 8G_1^{xx_1} \delta G^{xx_1} + 16\delta G^{xx_1} \delta G^{xx_1} + \{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}], \tag{B.2}$$

$$\Sigma_x^{x_1x_1}(\mathbf{p}) = \frac{1}{2} w^2 \frac{1}{N} \sum_{\mathbf{q}} [G_R^{xx} (5G_R^{xx} - 6G_L^{xx} - 4G_{LA}^{xx}) + G_L^{xx} G_L^{xx} + 8G_x^{x_1x_1} G_x^{x_1x_1} - 32G_1^{xx_1} \delta G^{xx_1} + 24\delta G^{xx_1} \delta G^{xx_1} + \{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}], \tag{B.3}$$

$$\Sigma_1^{xx_1}(\mathbf{p}) = w^2 \frac{1}{N} \sum_{\mathbf{q}} [(-G_R^{xx} + G_L^{xx} + 2G_x^{x_1x_1} - G_L^{x_1x_1} - 2G_{LA}^{x_1x_1}) G_1^{xx_1} + (-2G_R^{xx} + 2G_L^{xx} + 2G_{LA}^{xx} - G_x^{x_1x_1} - 6G_R^{x_1x_1} + 6G_L^{x_1x_1} + 5G_{LA}^{x_1x_1}) \delta G^{xx_1} + \{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}], \tag{B.4}$$

$$\delta \Sigma^{xx_1}(\mathbf{p}) = w^2 \frac{1}{N} \sum_{\mathbf{q}} [(-G_R^{xx} - 2G_L^{x_1x_1}) G_1^{xx_1} + (G_L^{xx} + 2G_x^{x_1x_1} + G_L^{x_1x_1} - 2G_{LA}^{x_1x_1}) \delta G^{xx_1} + \{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}], \tag{B.5}$$

$$\Sigma_L^{x_1x_1}(\mathbf{p}) = w^2 \frac{1}{N} \sum_{\mathbf{q}} [3G_R^{x_1x_1} G_R^{x_1x_1} - 4G_L^{x_1x_1} G_L^{x_1x_1} - 8G_L^{x_1x_1} G_{LA}^{x_1x_1} + \{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}], \tag{B.6}$$

$$\Sigma_{LA}^{x_1x_1}(\mathbf{p}) = \frac{1}{2} w^2 \frac{1}{N} \sum_{\mathbf{q}} [3G_R^{x_1x_1} G_R^{x_1x_1} + 21G_L^{x_1x_1} G_L^{x_1x_1} - 24G_R^{x_1x_1} G_L^{x_1x_1} - 8G_{LA}^{x_1x_1} G_{LA}^{x_1x_1} + 16G_L^{x_1x_1} G_{LA}^{x_1x_1} + \{\mathbf{q} \leftrightarrow \mathbf{p} - \mathbf{q}\}]. \tag{B.7}$$

Any of the self-energy components above has the following generic structure:

$$\Sigma(\mathbf{p}) = x_1(wq_1)[\tilde{C}_d \bar{\Lambda}^{2-\epsilon} + f(p^2/2wq_1) \bar{\Lambda}^{-\epsilon} + \sigma^{(a)}(p^2/2wq_1) + \sigma^{(na)}] \tag{B.8}$$

where $\sigma^{(a)}$ and $\sigma^{(na)}$ are analytical and nonanalytical contributions in x_1 , respectively, and both have corrections which are smaller by factors of x_1 (or higher integer powers of x_1) and, therefore, are irrelevant for the present calculation. While f and $\sigma^{(a)}$ have simple one-mass-scale momentum dependence, $\sigma^{(na)}$ has the double-mass-scale structure like the free propagator:

$$\sigma^{(na)}(p^2) = x_1^{-\frac{\epsilon}{2}+a} \begin{cases} f^{(l)}(p^2/2wq_1) & \text{for } p^2 \sim 2wq_1, \\ x_1^{-b} f^{(s)}(p^2/x_1 2wq_1) & \text{for } p^2 \sim x_1 2wq_1. \end{cases} \tag{B.9}$$

\tilde{C}_d is always zero except for Σ_R^{xx} , Σ_L^{xx} , and $\Sigma_L^{x_1 x_1}$; in these cases $\tilde{C}_d = -4C_d^{-1}/(d - 4)$ where $C_d \equiv 6\Gamma(d/2 - 2)\Gamma(4 - d/2)$. Some properties of the self-energy components which are relevant for the present calculation are summarized in Table 1.

Appendix C. Details of the different self-energy contributions to X_2

X_2 of Eq. (22) can be studied, and its relevant terms computed, using Eqs. (2), (20), (24), and the results of the preceding appendices, mainly (A.1), (A.3), (B.1) to (B.7), and Table 1.

- Σ_R^{xx} :

This replicon contribution can be conveniently evaluated by using (A.2) and writing $\Sigma_R^{xx}(\mathbf{p}) - \delta M = [\Sigma_R^{xx}(\mathbf{p}) - \Sigma_R^{xx}(\mathbf{p} = \mathbf{0})] + x_1 w q_1 r^2$. It is useful to add $-2m_c^{(2)} q_1 r$ with the first term in $m_c^{(2)}$, see (2), to the contribution with the zero-momentum subtraction to yield:

- (i) $\Sigma_R^{xx}(\mathbf{p}) - \Sigma_R^{xx}(\mathbf{p} = \mathbf{0})$ term:

$$\begin{aligned} X_2 = & -2C_d^{-1} r (wq_1^2) x_1^2 \int_0^{\bar{\Lambda}} dp p^{d-1} \\ & \times \left\{ 2G_R^3 \left[\sigma^{(a)}(p^2) + \frac{d-4}{3\epsilon} - r^2 \right. \right. \\ & + \left. \frac{4}{4-d} C_d^{-1} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)^2\Gamma(3-\frac{d}{2})}{\Gamma(d-2)} p^{2-\epsilon} \right] \\ & + (G_R^2 G_L^2 - 2r^2 G_R^3 G_L^2) \left[\sigma^{(a)}(p^2) + \frac{d-4}{3\epsilon} - r^2 \right. \\ & \left. \left. - \frac{4}{d} C_d^{-1} \frac{d-4}{\epsilon} \bar{\Lambda}^{-\epsilon} p^2 \right] \right\}, \end{aligned}$$

and X_2 takes the form for generic $6 < d < 8$:

$$X_2 = (wq_1^2) x_1^2 (A_d \bar{\Lambda}^{-2\epsilon} + A'_d \bar{\Lambda}^{-\epsilon} + A''_d) \tag{C.1}$$

with some dimension- and r -dependent amplitudes A_d , A'_d , and A''_d .⁵ In fact $A_d \sim r$, making the second derivative of X_2 proportional to $\bar{\Lambda}^{-\epsilon}$:

⁵ See (B.8) and Table 1 for notations and results. For the sake of avoiding complicated notations, we will not indicate the actual self-energy component in the quantities like $\sigma^{(a)}$, A_d , A'_d , etc., although they are different for different self-energy components listed here.

$$\frac{d^2}{dr^2} X_2 = (wq_1^2)x_1^2 \frac{d^2}{dr^2} (A'_d \bar{\Lambda}^{-\epsilon} + A''_d).$$

The integral is valid even in six dimensions where we get:

$$X_2 = (wq_1^2)x_1^2 \left(-\frac{7}{27}r \ln^2 \bar{\Lambda} + \frac{49}{162}r \ln \bar{\Lambda} + \frac{20}{27}r^3 \ln \bar{\Lambda} + O(1) \right).$$

(ii) The $x_1(wq_1)r^2$ term:

Inserting this term into (A.3), and after some manipulations with the propagators of Eq. (A.1), it follows:

$$\begin{aligned} X_2 &= -2C_d^{-1}r^3(wq_1^2)x_1^2 \int_0^{\bar{\Lambda}} dp p^{d-1} G_R G_L [2G_L + 5G_R G_L + 2(1-r^2)G_R^2 G_L] \\ &= (wq_1^2)x_1^2 (A'_d \bar{\Lambda}^{-\epsilon} + A''_d). \end{aligned}$$

The classification of the far infrared propagators in the end of Appendix A makes it possible to compute the contribution of the small mass regime $p^2 \sim x_1(2wq_1)$. One gets a dangerous term nonanalytical in x_1 , namely $X_2 \sim (wq_1^2)x_1^{2-\epsilon/2}$, which yields a $\ln x_1$ in six dimensions:

$$X_2 = (wq_1^2)x_1^2 \left[-\frac{2}{3}r^3 \ln \bar{\Lambda} + \frac{1}{3}r^3(1-r^2) \ln x_1 + O(1) \right], \quad d = 6.$$

Adding together the results of (i) and (ii), one finally gets the Σ_R^{xx} -insertion result in six dimensions:

$$\begin{aligned} X_2 &= (wq_1^2)x_1^2 \left[-\frac{7}{27}r \ln^2 \bar{\Lambda} + \frac{49}{162}r \ln \bar{\Lambda} + \frac{2}{27}r^3 \ln \bar{\Lambda} \right. \\ &\quad \left. + \frac{1}{3}r^3(1-r^2) \ln x_1 + O(1) \right]. \end{aligned} \tag{C.2}$$

• Σ_L^{xx} and $\Sigma_L^{x_1x_1}$:

The two longitudinal terms in the right hand side of Eq. (A.3) can be most conveniently written as

$$-\frac{1}{2}r G_R^2 G_L (\Sigma_L^{xx} - \Sigma_L^{x_1x_1}) + r G_R^2 G_L^2 [(1 - G_L) - r^2 G_L] (\Sigma_L^{x_1x_1} - \delta M).$$

(i) The $\Sigma_L^{xx} - \Sigma_L^{x_1x_1}$ part produces, due to the $\sigma^{(na)}$ in Σ_L^{xx} , a dangerous nonanalytical term in X_2 , but the $\bar{\Lambda}^{-2\epsilon}$ and $\bar{\Lambda}^{-\epsilon}$ contributions are canceled by the subtraction:

$$X_2 = (wq_1^2)x_1^2 (A''_d + B_d x_1^{-\epsilon/2}).$$

The interplay between the analytical and nonanalytical terms in x_1 [see Eqs. (B.8), (B.9), and the entries for Σ_L^{xx} in Table 1] produces the $\ln x_1$ for $d = 6$:

$$\begin{aligned} X_2 &= -2w(2wq_1)^{1-\epsilon/2} \frac{1}{N} \sum_{\mathbf{p}} \left[-\frac{1}{2}r G_R^2 G_L \right] \\ &\quad \times \left[-x_1(wq_1) \frac{2}{3}(1-r^2)^2 G_L (1-x_1^{-\epsilon/2}) \frac{1}{\epsilon} \right] \\ &\stackrel{d=6}{=} (wq_1^2)x_1^2 \left[-\frac{1}{18}r(1-r^2)^2 \ln x_1 + O(1) \right]. \end{aligned} \tag{C.3}$$

(ii) The $\Sigma_L^{x_1 x_1} - \delta M$ insertion yields the contribution to X_2 in a generic dimension just as in (C.1). In six dimensions, it becomes:

$$X_2 = (wq_1^2)x_1^2 \left(-\frac{2}{27}r \ln^2 \bar{\Lambda} - \frac{11}{81}r \ln \bar{\Lambda} + \frac{4}{27}r^3 \ln \bar{\Lambda} + O(1) \right).$$

Finally the complete six-dimensional result for the Σ_L^{xx} and $\Sigma_L^{x_1 x_1}$ insertions is the sum of (i) and (ii):

$$X_2 = (wq_1^2)x_1^2 \left[-\frac{2}{27}r \ln^2 \bar{\Lambda} - \frac{11}{81}r \ln \bar{\Lambda} + \frac{4}{27}r^3 \ln \bar{\Lambda} - \frac{1}{18}r(1-r^2)^2 \ln x_1 + O(1) \right]. \tag{C.4}$$

- $\Sigma_x^{x_1 x_1}$:

This self-energy is ultraviolet convergent, which is reflected by the fact that $f(p^2) \equiv 0$, see Table 1. In fact, the whole two-loop graph built up from this self-energy is finite for $\Lambda \rightarrow \infty$, and there is no $\ln \bar{\Lambda}$ in six dimensions. The leading infrared contribution is, however, exactly the same as in the case of the longitudinal self-energy, i.e. (C.3): see Eq. (A.3) and the entries in Table 1. We thus finally have:

$$X_2 = (wq_1^2)x_1^2 \left[-\frac{1}{18}r(1-r^2)^2 \ln x_1 + O(1) \right], \quad d = 6. \tag{C.5}$$

- $\Sigma_1^{xx_1}$:

This term — which is somewhat complicated, but manageable when we are looking for the logarithms in six dimensions — must be treated together with the second part of the $-2m_c^{(2)}q_1 r$ subtraction, see Eq. (2). In generic dimensions d it has the structure of Eq. (C.1) together with a dangerous nonanalytical contribution $X_2 \sim (wq_1^2)x_1^{2-\epsilon/2}$. Although the $\bar{\Lambda}^{-2\epsilon}$ term suggests that a $\ln^2 \bar{\Lambda}$ should exist in $d = 6$, the two such terms cancel out each other. Finally we have:

$$X_2 = (wq_1^2)x_1^2 \left[\frac{5}{9}r \ln \bar{\Lambda} - \frac{1}{9}r^3 \ln \bar{\Lambda} - \frac{1}{18}r(1-r^2)^2 \ln x_1 + O(1) \right], \quad d = 6. \tag{C.6}$$

- $\delta \Sigma^{xx_1}$:

For generic d , X_2 takes the form of (C.1), and there is no dangerous nonanalytic correction. One can relatively easily find:

$$X_2 = (wq_1^2)x_1^2 \left[\frac{2}{9}r \ln^2 \bar{\Lambda} - \frac{1}{3}r^3 \ln \bar{\Lambda} + O(1) \right], \quad d = 6. \tag{C.7}$$

- $\Sigma_{LA}^{x_1 x_1}$:

We have again an X_2 like in Eq. (C.1) without any dangerous nonanalytic correction. The six-dimensional limit yields

$$X_2 = (wq_1^2)x_1^2 \left[\frac{1}{9}r \ln^2 \bar{\Lambda} - \frac{1}{18}r \ln \bar{\Lambda} + O(1) \right], \quad d = 6. \tag{C.8}$$

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