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Symmetry preserving regularization with a cutoff

Research Article

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Abstract: A Lorentz and gauge symmetry preserving regularization method is proposed in 4 dimensions based on a momentum cutoff. We use the conditions of gauge invariance or equivalently the freedom to shift the loop momentum to define the evaluation of the terms carrying even number of Lorentz indices, e.g. proportional to $k_\mu k_\nu$. The remaining scalar integrals are calculated with a four dimensional momentum cutoff. The finite terms (independent of the cutoff) are free of ambiguities coming from subtractions in non-trivial cases. Finite parts of the result are equal to that of dimensional regularization.

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1. Introduction

Several regularization methods are known and used in quantum field theory: three and four dimensional momentum cutoff, Pauli-Villars type, dimensional regularization, lattice regularization, Schwinger's proper time method and others directly linked to renormalization like differential renormalization. Dimensional regularization (DREG) [1] is the most popular and most appreciated as it respects the gauge and Lorentz symmetries. However DREG is not useful in all cases, for example it is not directly applicable to supersymmetric gauge theories as it modifies the number of bosons and fermions differently. DREG eliminates (does not identify) naive quadratic divergences, which may be important in low energy effective theories or in the renormalization group method of Wilson. Another shortcoming is that together with (modified) minimal subtraction DREG is a "mass independent" scheme, particle thresholds and decoupling are put in the theory by hand [2]. The choice of the ultraviolet regulator always depends on the problem.

In low energy effective field theories there is an explicit cutoff, with a well defined physical meaning. The cutoff gives the range of the validity of the model. There are few implementations in four dimensional theories: sharp momentum cutoff in three and four dimensions, modified operator regularization (based on Schwinger proper time method [3]). In the Nambu-Jona-Lasinio model different regularizations proved to be useful in calculating different physical quantities [4].

Using a naive momentum cutoff the symmetries are badly violated. The calculation of the QED vacuum polarization function ($\Pi_{\mu\nu}(q)$) exemplifies the problems. The Ward identity tells us that $q^\mu \Pi_{\mu\nu}(q) = 0$, e.g. in

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$$\Pi_{\mu\nu}(q) = q_\mu q_\nu \Pi_L(q^2) - g_{\mu\nu} q^2 \Pi_T(q^2) \tag{1}$$

$\Pi_L(q^2)$ and $\Pi_T(q^2)$ are equal to each other (denoted by $\Pi(q^2)$). Usually the condition $\Pi(0) = 0$ is required to define a subtraction to keep the photon massless at 1-loop. However this condition is ambiguous when for $q^2 \neq 0$ in QED or in more general models. For example in the case of two different masses in the loop, the condition simply fixes $\Pi(q^2, m_1, m_2)$ in the limit of degenerate masses at $q^2 = 0$. Ad hoc subtractions does not necessarily give satisfactory results.

There have been several proposals how to define a symmetry preserving cutoff regularization. The usual way is to start with a regularization which respects symmetries and find the connection using the momentum cutoff. In case of dimensional regularization Veltman already observed [5] that the naive quadratic divergences can be identified with the poles in two dimensions ($d = 2$) besides the usual logarithmic singularities in $d = 4$. This idea turned out to be fruitful. Hagiwara et al. [6] calculated electroweak radiative corrections originating from effective dimension-six operators, and later Harada and Yamawaki performed the Wilsonian renormalization group inspired matching of effective hadronic field theories [7]. Based on Schwinger's proper time approach Oleszczuk proposed the operator regularization method [8], and showed that it can be formulated as a smooth momentum cutoff respecting gauge symmetries [8, 9]. A momentum cutoff is defined in the proper time approach in [10] with the identification under loop integrals

$$k_\mu k_\nu \to \frac{1}{d} g_{\mu\nu} k^2 \tag{2}$$

instead[1] of the standard $d = 4$. The degree of the divergence determines d in the result: Λ^2 corresponds to $d = 2$ and $\ln(\Lambda^2)$ to $d = 4$. This way the authors correctly obtain the *divergent parts*, as they checked them for the QED vacuum polarization function and for the phenomenological chiral model.

Various authors formulated consistency conditions to maintain gauge invariance in the evaluation of divergent loop integrals. When finite [11] or infinite [12, 13] number of new regulator terms added to the propagators a'la Pauli-Villars, the integrals have at most logarithmic singularities and become tractable. The Pauli-Villars regularization technique was applied with subtractions to gauge invariant and chiral models [14–17]. Differential renormalization can be modified to fulfill consistency conditions automatically, a method known as constrained differential renormalization [18]. Another approach, later proved to be equivalent with the previous one [19], is the method of implicit regularization, in which a recursive identity (similar to Taylor expansion) is applied and all the dependence on the external momentum (q) is transferred to finite integrals. The divergent integrals contain only the loop momentum, thus universal local counter terms can cancel the potentially dangerous symmetry violating contributions [20, 21]. Gauge invariant regularization is implemented in the exact renormalization group method providing a cutoff without gauge fixing in [23]. Introducing a multiplicative regulator in the d-dimensional integral, the integrals can be calculated in the original dimension with the tools of DREG [24].

In this paper we give a definite method in four dimensions to use a well defined momentum cutoff. We show that there is a difficulty between naive application of Lorentz symmetry and gauge invariance. The core of the problem is that the contraction with $g^{\mu\nu}$ cannot necessarily be interchanged with the integration in divergent cases. The proper handling of the $k_\mu k_\nu$ terms in divergent loop-integrals solves the problems of momentum cutoff regularizations. This is the new result of the paper. Working in strictly four dimensions we use the conditions of respecting symmetries to define the integrals with free Lorentz indices. Using our method loop calculations can be reduced to scalar integrals and those can be evaluated with a sharp momentum cutoff. We give a simple and well defined algorithm to have unambiguous finite and infinite terms. The results respect gauge (chiral and other) symmetries and the finite terms agree with the result of DREG.

In Section 2 we present the definition of a momentum cutoff using the method of DREG, then we give the gauge symmetry preserving conditions which emerge from the calculation of the vacuum polarization amplitude. In Section 4 we discuss the condition of independence of momentum routing in loop diagrams. Section 5 shows that gauge invariance and freedom of shift in the loop momentum have the same origin. Next we show that the conditions are related to vanishing surface terms. In Section 7 we give a definition of the new regularization method and in Section 8 as an example we present the calculation of a general vacuum polarization function at 1-loop. In Section 9 we show that the QED Ward-Takahashi identity holds at finite order using the new method. Subsequently we conclude our work.

[1] In what follows we denote the metric tensor by $g_{\mu\nu}$ both in Minkowski and Euclidean space.

2. Momentum cutoff via dimensional regularization

DREG is very efficient and popular, because it preserves gauge and Lorentz symmetries. Performing standard steps the integrals are evaluated in $d=4-2\epsilon$ dimension. Generally the loop momentum integral is Wick rotated and the denominators are combined with a Feynman parameter (x). Subsequently the order of x and momentum integrals are changed. Shifting the loop momentum does not generate surface terms and it leads to a spherically symmetric denominator, terms linear in the momentum are dropped and (2) is used. Singularities are identified as $\frac{1}{\epsilon}$ poles, naive power counting shows that these are the logarithmic divergences of the theory.[2] In DREG quadratic or higher divergences are set identically to zero. However, Veltman noticed [5] that quadratic divergences can be calculated in $d=2-2(\epsilon-1)$ in the limit $\epsilon\to 1$. This observation led to a cutoff regularization based on DREG.

Carefully calculating the one and two point Passarino-Veltman functions in DREG and for the 4-momentum cutoff the divergences can be matched as [6, 7]

$$4\pi\mu^2\left(\frac{1}{\epsilon-1}+1\right)=\Lambda^2, \tag{3}$$

$$\frac{1}{\epsilon}-\gamma_E+\ln\left(4\pi\mu^2\right)+1=\ln\Lambda^2, \tag{4}$$

where μ is the mass-scale of dimensional regularization. The finite part of a divergent quantity is defined as

$$f_{\text{finite}}=\lim_{\epsilon\to 0}\left[f(\epsilon)-R(0)\left(\frac{1}{\epsilon}-\gamma_E+\ln 4\pi+1\right)-R(1)\left(\frac{1}{\epsilon-1}+1\right)\right], \tag{5}$$

where $R(0)$, $R(1)$ are the residues of the poles at $\epsilon=0,1$ respectively. Note that in the usual $\epsilon\to 0$ limit the left hand side (LHS) of (3) vanishes and no quadratic divergence appears in the original DREG.

The identifications above define a momentum cutoff calculation based on the symmetry preserving DREG formulae. This cutoff regularization is well defined, but still relies on DREG. Let us see the main properties in the calculation of the vacuum polarization function. In $\Pi_{\mu\nu}$ the quadratic divergence is partly originating from a $k_\mu k_\nu$ term via

$$\frac{1}{d}\cdot g_{\mu\nu}k^2,$$

which is evaluated at $d=2$ instead of the $d=4$ in the naive cutoff calculation. The Λ^2 terms cancel if and only if this term is evaluated at $d=2$. This is a warning that the usual

$$k_\mu k_\nu\to\frac{1}{4}g_{\mu\nu}k^2$$

substitution during the naive cutoff calculation of divergent integrals might be too naive, especially as an intermediate step, the Wick rotation is legal only for finite integrals. A further finite term additional to the logarithmic singularity results from the well known expansion in

$$\frac{1}{4-2\epsilon}\frac{1}{\epsilon}\simeq\frac{1}{4}\left(\frac{1}{\epsilon}+\frac{1}{2}\right),$$

and it is essential to retain gauge invariance. We stress that the shift of the loop momentum is allowed in DREG, hence this property should be inherited by an improved cutoff regularization. In the next sections we derive consistency conditions for general regularizations.

3. Consistency conditions – gauge invariance

Calculation in a gauge theory ought to preserve gauge symmetries. Consider the QED vacuum polarization function with massive electrons. We start generally (see Fig. 1) with two fermions with different masses in the loop [26] and restrict it to QED later,

$$i\Pi_{\mu\nu}(q)=-(-ig)^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_\mu\frac{\not k+m_a}{k^2-m_a^2}\gamma_\nu\frac{\not k+\not q+m_b}{(k+q)^2-m_b^2}\right). \tag{6}$$

$\Pi_{\mu\nu}$ is calculated with the standard technique, only the $k_\mu k_\nu$ terms are considered with care. After performing the trace, Wick rotating and introducing the Feynman x-parameter the loop momentum is shifted $(k_{E\mu}+xq_{E\mu})\to l_{E\mu}$,

[2] Similar identification can be done in three dimensional integrals [25].

$$\Pi_{\mu\nu} = g^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{2l_{E\mu}l_{E\nu} - g_{\mu\nu}\left(l_E^2 + \Delta\right) - 2x(1-x)q_{E\mu}q_{E\nu} + 2x(1-x)g_{\mu\nu}q_E^2}{\left(l_E^2+\Delta\right)^2}, \quad (7)$$

where $\Delta = x(1-x)q_E^2 + (1-x)m_a^2 + xm_b^2$. In QED $m_a = m_b = m$ and $g = e$ it simplifies to $\Delta_1 = x(1-x)q_E^2 + m^2$. Having a symmetric denominator and symmetric volume of integration the terms linear in $l_{E\mu}$ are dropped. After changing the order of momentum- and x-integration the loop momentum is shifted by x-dependent values, $xq_{E\mu}$ and sum up the results during the integration. Different shifts sums up to a meaningful result only if the shift does not modify the value of the momentum integral (this will be discussed in the next section).

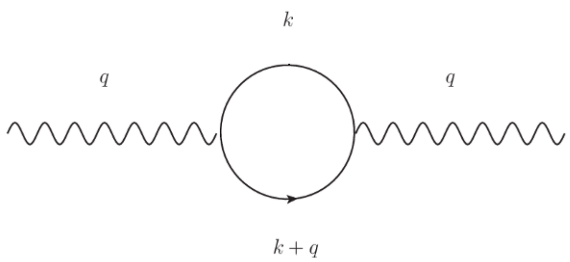

Figure 1. 1-loop vacuum polarization diagram.

In QED the Ward identity tells us, that

$$q^\mu \Pi_{\mu\nu}(q) = 0. \quad (8)$$

In (7) the terms proportional to q_E fulfill the Ward-identity (8) and what remains is the condition of gauge invariance

$$\int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_{E\mu}l_{E\nu}}{\left(l_E^2+\Delta_1\right)^2} = \frac{1}{2} g_{\mu\nu} \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{\left(l_E^2+\Delta_1\right)}. \quad (9)$$

This condition appeared already in [13, 20]. Any gauge invariant regulator should fulfill (9). It holds in dimensional regularization and in the momentum cutoff based on DREG of Section 2. In [11, 13] a similar relation defined the finite or infinite Pauli-Villars terms to maintain gauge invariance.
So far the x integrals have not been performed. Expanding the denominator in q^2, the x-integration can be done easily and we arrive at a condition for gauge invariance at each order of q^2. At order q^{2n} we obtain (omitting the factor $(2\pi)^4$)

$$\int d^4 l_E \frac{l_{E\mu}l_{E\nu}}{\left(l_E^2+m^2\right)^{n+1}} = \frac{1}{2n} g_{\mu\nu} \int d^4 l_E \frac{1}{\left(l_E^2+m^2\right)^n}, \quad n = 1, 2, \ldots . \quad (10)$$

The conditions (10) are valid for arbitrary m^2 mass, so it holds for any function Δ independent of the loop momentum in 1-loop two or n-point functions with arbitrary masses in the propagators. These conditions mean that in any gauge invariant regularization the two sides of (10) should give the same result. We will use this condition to define the LHS of (10) in the new improved cutoff regularization. This is the novelty of our regularization method.

4. Consistency conditions – momentum routing

Evaluating any loops in QFT one encounters the problem of momentum routing. The choice of the internal momenta should not affect the result of the loop calculation. The simplest example is the 2-point function. In (6) there is a loop momentum k, and the external momentum q (see Fig. 1) is put on one line $(k+q, k)$, but any partition of the external momentum $(k+q+p, k+p)$ must be as good as the original. The arbitrary shift of the loop momentum should not change the physics. This independence of the choice of the internal momentum gives a conditions. We will impose it on a very simple loop integral

$$\int d^4 k \frac{k_\mu}{k^2 - m^2} - \int d^4 k \frac{k_\mu + p_\mu}{(k+p)^2 - m^2} = 0, \quad (11)$$

which turns up during the calculation of the 2-point function. Expanding (11) in powers of p we obtain a series of condition, meaningful at $p, p^3, p^5 \ldots$. At linear order we arrive at

$$\int d^4 k \left(\frac{p_\mu}{k^2 - m^2} - 2\frac{k_\mu k \cdot p}{(k^2 - m^2)^2} \right) = 0, \quad (12)$$

which is equivalent to (10) for $n = 1$. At order p^3 the linear combination of two conditions should vanish

$$p_\rho p_\alpha p_\beta \int d^4k \left[\left(\frac{4k_\alpha k_\beta}{(k^2-m^2)^3} - \frac{g_{\alpha\beta}}{(k^2-m^2)^2} \right) g_{\mu\rho} -4k_\mu \left(\frac{2k_\alpha k_\beta k_\rho}{(k^2-m^2)^4} - \frac{g_{\alpha\beta}k_\rho}{(k^2-m^2)^3} \right) \right] = 0. \tag{13}$$

These two conditions become separated if the freedom of the shift of the loop momentum is considered in

$$\int d^4k \frac{k_\mu}{(k^2-m^2)^2}.$$

At leading order it provides

$$p_\nu \int d^4k \left(\frac{g_{\mu\nu}}{(k^2-m^2)^2} - 4\frac{k_\mu k_\nu}{(k^2-m^2)^3} \right) = 0, \tag{14}$$

equivalent with (10) for $n = 2$. Using (14) twice the second part of the condition (13) connects 4 loop momenta numerators to 2 k's. Symmetrizing the indices we obtain

$$\int d^4k \frac{k_\alpha k_\beta k_\mu k_\rho}{(k^2-m^2)^4} = \frac{1}{24} \int d^4k \frac{g_{\alpha\beta}g_{\mu\rho} + g_{\alpha\mu}g_{\beta\rho} + g_{\alpha\rho}g_{\beta\mu}}{(k^2-m^2)^2}. \tag{15}$$

Invariance of momentum routing provides conditions for symmetry preserving regularization and these conditions are equivalent with the conditions coming from gauge invariance.

5. Gauge invariance and loop momentum shift

We show at the one loop level that gauge invariance of the vacuum polarization function is equivalent to invariance of a special loop integrand to shifting the loop momentum (11). Consider $\Pi_{\mu\nu}$ defined in (6), performing the trace we obtain

$$i\Pi_{\mu\nu}(q) = -g^2 \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu(k_\nu+q_\nu) + k_\nu(k_\mu+q_\mu) - g_{\mu\nu}\left(k^2 + k\cdot q - m_a m_b\right)}{(k^2-m_a^2)\left((k+q)^2-m_b^2\right)}. \tag{16}$$

In particular in QED $m_a = m_b = m$, gauge invariance requires (8), which simplifies to

$$iq^\nu \Pi_{\mu\nu}(q) = g^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{k_\mu+q_\mu}{((k+q)^2-m^2)} - \frac{k_\mu}{(k^2-m^2)} \right) = 0. \tag{17}$$

This example shows that the Ward identity is fulfilled only if the shift of the loop momentum does not change the value of the integral, as in (11).

In [21] based on the general diagrammatic proof of gauge invariance it is shown that the Ward identity is fulfilled if the difference of a general n-point loop and its shifted version vanishes

$$-i \int d^4p_1 Tr \left[\frac{i}{\not p_n - m}\gamma^{\mu_n} \dots \frac{i}{\not p_1 - m}\gamma^{\mu_1} - \frac{i}{\not p_n + \not q - m}\gamma^{\mu_n} \dots \frac{i}{\not p_1 + \not q - m}\gamma^{\mu_1} \right] = 0. \tag{18}$$

We interpret (17) and (18) as a necessary condition for gauge invariant regularizations.

6. Consistency conditions – vanishing surface terms

All the previous conditions are related to the volume integral of the total derivative

$$\int d^4k \frac{\partial}{\partial k^\nu} \left(\frac{k_\mu}{(k^2+m^2)^n} \right) = \int d^4k \left(\frac{k_\mu k_\nu}{(k^2+m^2)^{n+1}} - \frac{1}{2n} g_{\mu\nu} \frac{1}{(k^2+m^2)^n} \right), \quad n = 1, 2, \dots . \tag{19}$$

The total derivative on the LHS leads to surface terms [22], which vanish for integrals with finite value and should vanish for symmetry preserving regularization. In our improved regularization this will follow from new definitions. The LHS is in connection with an infinitesimal shift of the loop momentum k, it should be zero if the integral of the term in the delimiter is invariant to the shift of the loop momentum. The vanishing of this surface term reproduces the previous conditions (12) and (10) on the RHS. In (19) starting with an odd number of k's in the numerator we end up with some conditions, three k's for $n = 3$ provide (15) after some algebra. Starting with even number of k_μ's in the numerator on the LHS in (19) we obtain relations between odd number of k_μ's in the numerators, which vanish separately.

These surface terms all vanish in DREG and give the basis of DREG respecting Lorentz and gauge symmetries. Vanishing of the surface term is inherent in any regularization, such as the improved momentum cutoff, if the identification (9) is understood to evaluate integrals involving an even number of free Lorentz indices, e.g. numerators such as $k_\mu k_\nu$. The value of integrals with odd number of k's in the numerator are similarly dictated by symmetry, these are required to vanish by the symmetry of the integration volume.

7. Improved momentum cutoff regularization

We propose a new symmetry preserving regularization based on a 4-dimensional momentum cutoff. For this improved momentum cutoff regularization method a simple sharp momentum cutoff is introduced to calculate the divergent scalar integrals in the end. The evaluation of loop-integrals starts with the usual Wick rotation, Feynman parametrization and loop momentum shift. The only crucial modification is that the potentially symmetry violating loop integrals containing the loop momenta with free Lorentz indices are calculated explicitly with the identification

$$\int \frac{l_{E\mu} l_{E\nu}}{\left(l_E^2+\Delta\right)^{n+1}} \to \frac{1}{2n} g_{\mu\nu} \int \frac{1}{\left(l_E^2+\Delta\right)^{n}} \tag{20}$$

under the loop integrals or with more momenta using the condition (15) or generalizations of it, e.g.

$$\int \frac{l_{E\mu} l_{E\nu} l_{E\rho} l_{E\sigma}}{\left(l_E^2+\Delta\right)^{n+1}} \to \frac{g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}}{4n(n-1)} \cdot \int \frac{1}{\left(l_E^2+\Delta\right)^{n-1}}. \tag{21}$$

The momentum integrals containing further the loop momentum with summed indices (e.g. l_E^2) in the numerator are simplified in the standard way cancelling a factor in the denominator

$$\int \frac{l_E^2\, l_{E\mu} l_{E\nu}\ldots}{\left(l_E^2+\Delta\right)^{n+1}} = \int \frac{l_{E\mu} l_{E\nu}\ldots}{\left(l_E^2+\Delta\right)^{n}} - \int \frac{\Delta\, l_{E\mu} l_{E\nu}\ldots}{\left(l_E^2+\Delta\right)^{n+1}}. \tag{22}$$

Integrals with odd number of loop momenta vanish identically. These identifications guarantee gauge invariance and freedom of shift in the loop momentum. Under any regularized momentum integrals the identifications (20) or generalizations such as (21) are understood as a part of the regularization procedure for $n = 1, 2, \ldots$. For finite integrals (non divergent, for high enough n) the standard calculation automatically fulfills (20), (21). The connection with the standard substitution of free indices is discussed in Appendix A.

Fulfilling the condition (10) via the substitution (20) the results of the momentum cutoff based on DREG of Section 2 are completely reproduced performing the calculation in the physical dimensions $d = 4$ [26, 27]. The next two examples show that the new regularization provides a robust framework for calculating loop integrals and respects symmetries.

8. Vacuum polarization function

As an example let us calculate the vacuum polarization function of Fig. 1 in a general gauge theory with fermion masses m_a, m_b. Performing the calculation in 4 dimensions generally the Ward identities (required by the theory) are restored by ambiguous and ad hoc subtractions. The finite terms of different calculations do not match each other in the literature, see [28], papers citing it and [29]. For sake of simplicity we consider only vector couplings. Performing the trace in (6) we obtain (16). Now we can introduce a Feynman x-parameter, shift the loop momentum and obtain (7) after dropping the linear terms. Generally we are interested in low energy observables like the precision electroweak parameters and need the first few terms in the power series of $\Pi_{\mu\nu}(q)$. Using the rule (20) for $n = 1$ and expanding the denominator in q^2 the scalar loop and x-integrals can be easily calculated with a 4-dimensional momentum cutoff (Λ). The result in this construction is automatically transverse

$$\Pi_{\mu\nu}(q) = \frac{g^2}{4\pi^2}\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\left[\Pi(0) + q^2\Pi'(0) + \ldots\right]. \tag{23}$$

The terms independent of the cutoff completely agree with the results of DREG [27] the logarithmic singularity can be matched with the $\frac{1}{\epsilon}$ terms using (4). Up to $\mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)$ we obtain

$$\Pi(0) = \frac{1}{4}(m_a^2 + m_b^2) - \frac{1}{2}(m_a - m_b)^2 \ln\left(\frac{\Lambda^2}{m_a m_b}\right) - \frac{m_a^4 + m_b^4 - 2m_a m_b\left(m_a^2 + m_b^2\right)}{4\left(m_a^2 - m_b^2\right)} \ln\left(\frac{m_b^2}{m_a^2}\right). \tag{24}$$

The first derivative is

$$\begin{aligned}\Pi'(0) = &- \frac{2}{9} - \frac{4m_a^2 m_b^2 - 3m_a m_b\left(m_a^2 + m_b^2\right)}{6\left(m_a^2 - m_b^2\right)^2} + \frac{1}{3}\ln\left(\frac{\Lambda^2}{m_a m_b}\right) \\ &+ \frac{\left(m_a^2 + m_b^2\right)\left(m_a^4 - 4m_a^2 m_b^2 + m_b^4\right) + 6m_a^3 m_b^3}{6\left(m_a^2 - m_b^2\right)^3} \ln\left(\frac{m_b^2}{m_a^2}\right).\end{aligned} \tag{25}$$

The photon remains massless in QED, as in the limit, $m_a = m_b$ we obtain $\Pi(0) = 0$.
The proposed regularization is robust and gives the same result if the calculation is organized in a different way. Introducing Feynman parameters and shifting the loop momentum can be avoided if we need only the first few terms in the Taylor expansion of q. For small q the second denominator in (16) can be Taylor expanded, for simplicity we give the expanded integrand for equal masses, up to $O(q^4)$

$$\begin{aligned}\Pi_{\mu\nu}(q) = &- g^2 \int \frac{d^4k_E}{(2\pi)^4}\Bigg[2k_\mu k_\nu\left(\frac{1}{\left(k_E^2+m^2\right)^2} - \frac{q_E^2}{\left(k_E^2+m^2\right)^3} + \frac{4\left(k_E\cdot q_E\right)^2}{\left(k_E^2+m^2\right)^4}\right) \\ &- \frac{2\left(k_{E\mu}q_{E\nu} + k_{E\nu}q_{E\mu}\right)k_E\cdot q_E}{\left(k_E^2+m^2\right)^3} - g_{\mu\nu}\left(\frac{1}{\left(k_E^2+m^2\right)^2} - \frac{q_E^2}{\left(k_E^2+m^2\right)^3} + \frac{2\left(k_E\cdot q_E\right)^2}{\left(k_E^2+m^2\right)^4}\right)\Bigg].\end{aligned} \tag{26}$$

Taking into account that $k_E \cdot q_E = k_{E\alpha} q_{E\alpha}$, (20) and (21) can be used and the remaining scalar integrals can be easily calculated. The result agrees with (24) and (25) and the finite terms with DREG if and only if we use the proposed symmetry preserving substitutions. Applying the naive $k_{E\mu}k_{E\nu} \to \frac{1}{4} g_{\mu\nu} k_E^2$ substitution in both approaches the finite terms will differ from each other and also from the result of DREG. This is why finite terms differ from each other in [28] and [29].
The calculation of the $\Pi_{\mu\nu}$ function at 1-loop shows that the new regularization gives a robust gauge invariant result and the finite terms agree with DREG.

9. The Ward-Takahashi identity

In this section we show with explicit calculation that the QED Ward-Takahashi identity is fulfilled for infinite and finite terms using the proposed regularization at 1-loop. Following the notation of [30] it has to be proved that

$$\left.\frac{d\Sigma}{dp_\mu}\right|_{\not p = m} = -\,\delta\Gamma^\mu(p,p)\Big|_{\not p = m}, \tag{27}$$

where Σ is the electron self-energy (see Fig. 2 left panel)

$$\begin{aligned}&-i\bar{u}(p)\Sigma u(p) \\ &\quad = -e^2\int_0^1 dx \int \frac{d^4l}{(2\pi)^4}\bar{u}(p)\frac{-2x\,\not p + 4m}{\left(l^2 - \Delta_2 + i\epsilon\right)^2}u(p),\end{aligned} \tag{28}$$

here $\Delta_2 = -x(1-x)p^2 + (1-x)m^2 + x\mu^2$, $l = k - xp$, m is the mass of the electron and μ is the infrared regulator.

$$\left.\frac{d\Sigma}{dp_\mu}\right|_{\not p = m} = \frac{\alpha}{2\pi}\int_0^1 dx\left[-x\gamma^\mu\left(\ln\left(\frac{\Lambda^2}{(1-x)^2m^2 + x\mu^2}\right) - 1 + \frac{2(2-x)(1-x)}{(1-x)^2m^2 + x\mu^2}\right)\right], \tag{29}$$

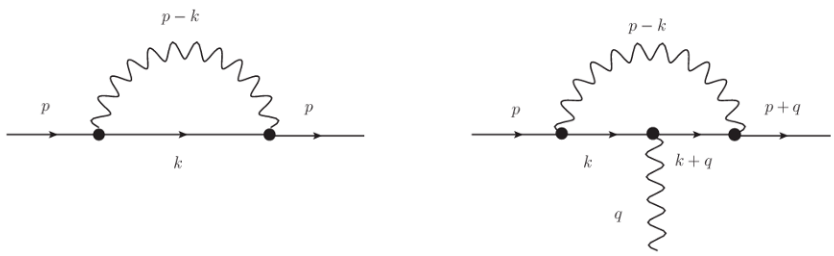

Figure 2. 1-loop diagrams for the Ward identity.

$\delta\Gamma^\mu$ is the electron vertex correction (see Fig. 2 right panel)

$$\bar{u}(p')\delta\Gamma^\mu u(p) = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^\mu(\not{k}+\not{q}) + m^2\gamma^\mu - 2m(k+(k+q))^\mu\right]u(p)}{((k-p)^2+i\epsilon)\,((k+q)^2-m^2+i\epsilon)\,(k^2-m^2+i\epsilon)}. \tag{30}$$

After using the Dirac equation in the limit $p = p'$ and $q = 0$ we obtain

$$-i\bar{u}(p)\delta\Gamma^\mu u(p) = 2e^2 \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^4l}{(2\pi)^4} \times \frac{\bar{u}(p)\left[\not{l}\gamma^\mu \not{l} + (z^2-4z+1)m^2\gamma^\mu\right]u(p)}{(l^2-\Delta_3+i\epsilon)^3}, \tag{31}$$

where $\Delta_3 = (1-z^2)m^2 + z\mu^2$ and $l = k - zp$. Here $\not{l}\gamma^\mu\not{l} = 2l^\mu l^\nu\gamma_\nu - \gamma^\mu l^2$, for the first term (20) should be used for $n = 2$ or directly (B6) from Appendix B. After the momentum and x, y integration

$$\delta\Gamma^\mu|_{\not{p}=m} = \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-z)\left(\ln\left(\frac{\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) - 1 + \frac{(1-4z+z^2)}{(1-z)m^2+z\mu^2}\right)\right]. \tag{32}$$

The result of the new method is the constant -1 after the log, with the naive calculation using (A1) one would obtain $-\frac{1}{2}$. Calculating the Feynman-parameter integral taking care of the infrared regulator the identity (27) is valid up to $\frac{m^2}{\Lambda^2}$ terms at 1-loop

$$-\left.\frac{d\Sigma}{dp_\mu}\right|_{\not{p}=m} = \delta\Gamma^\mu(p,p)\left.\vphantom{\frac{d\Sigma}{dp_\mu}}\right|_{\not{p}=m} = \frac{\alpha}{2\pi}\left(\frac{1}{2}\ln\left(\frac{\Lambda^2}{m^2}\right) + \ln\left(\frac{\mu^2}{m^2}\right) + 2\right) + \mathcal{O}\left(\frac{m^2}{\Lambda^2}, \frac{\mu^2}{\Lambda^2}\right). \tag{33}$$

We have seen that the proposed method provides regularized 1-loop electron self-energy and vertex correction in QED which fulfill the Ward-Takahashi identity.

10. Conclusions

In this paper we have presented a new method for the reliable calculation of divergent 1-loop diagrams (not involving γ_5) with a four dimensional momentum cutoff. Various conditions were derived to maintain gauge symmetry, to have the freedom of momentum routing or shifting the loop momentum. These conditions were known by several authors [11, 13, 20, 21]. Our new proposal is that these conditions will be satisfied for the regularization process if terms proportional to loop momenta with an even number of free Lorentz indices (e.g. $\sim k_\mu k_\nu$) are calculated according to the special rules (20) and (21) or generalizations thereof. In the end the scalar integrals are calculated with a simple momentum cutoff. The calculation is robust – at least at 1-loop level – as we have shown via the fermionic contribution to the vacuum polarization function. The finite terms agree with the ones from DREG in all examples. The connection with DREG is more transparent if one uses alternatively the $k_\mu k_\nu \to \frac{1}{d} g_{\mu\nu} k^2$ or (A6) substitution and d takes different values determined by the degree of divergence in each term (A3), (A4), (A5). We stress that this new regularization holds without DREG as the substitutions (20), (21) and scalar integration with a cutoff are independent of DREG. The success of both regularizations based on the property that they fulfill the consistency conditions of gauge invariance and momentum shifting.

At 1-loop the finite terms in the improved momentum cutoff are found to be equivalent with DREG. For practical calculations we propose to use the same renormalization scheme, the MS or $\overline{MS}$ subtractions with the BPHZ forest formula as with DREG. DREG is not just the generally used method, but it is proved to be a mathematically rigorous regularization within the Epstein-Glaser framework [31, 32]. The equivalence of the results of the proposed method and DREG gives a hint that the improved cutoff method with e.g. MS subtraction and BPHZ can be used as a renormalization scheme for more complicated diagrams.

Regularization schemes based on consistency conditions have been applied to more involved cases. Differential renormalization fulfilling similar conditions is proved to be equivalent to regularization within the Epstein-Glaser framework [33]. Constrained differential renormalization is useful in supersymmetric [34] and non-Abelian gauge theories, it fulfills Slavnov-Taylor identities at one and two loops [35]. Implicit regularization [20, 21] requires the same conditions as we used and it was successfully applied to the Nambu-Jona-Lasinio model [20] and to higher loop calculations in gauge theory. It was shown that the conditions guarantee gauge invariance generally and the Ward identities are fulfilled explicitly in QED at two-loop order [21]. In an effective composite Higgs model, the Fermion Condensate Model [36] oblique radiative corrections (S and T parameters) were calculated in DREG and with the improved cutoff, too, the finite results completely agree. The calculation involved vacuum polarization functions with two different fermion masses and no ambiguity appeared [26, 27].

The new regularization is advantageous in special loop-calculations where one wants to keep the cutoff of the model, like in effective theories, derivation of renormalization group equations, extra dimensional scenarios or in models explicitly dependent on the space-time dimensions, like supersymmetric theories. We argue that the method can be successfully used in higher order calculations containing terms up to quadratic divergences in (non-Abelian) gauge theories, as it allows for shifts in the loop momenta, which guarantees the 't Hooft identity [21, 37]. This symmetry preserving method can be used also in automatized calculations (similar to [38]) as even the Veltman-Passarino functions [39] can be defined with the improved cutoff. The calculation of the famous triangle anomaly with the proposed method needs special care and treatment [40].

Appendix A: CONNECTION WITH THE STANDARD TECHNIQUE

What is the relation of the new method to the standard (textbook)

$$k_\mu k_\nu \to \frac{1}{4} g_{\mu\nu} k^2 \tag{A1}$$

substitution? We have to modify it in case of divergent integrals to respect gauge symmetry, i.e to fulfill (10). Lorentz invariance dictates that in (10) the LHS must be proportional to the only available tensor $g_{\mu\nu}$, i.e.

$$l_{E\mu} l_{E\nu} \to \frac{1}{d} g_{\mu\nu} l_E^2 \tag{A2}$$

can be used, where d is a number to be determined.[3] Now both sides of equation (10) can be calculated with a simple 4-dimensional momentum cutoff. The different powers of Λ can be matched on the two sides, and for $n = 1$ we obtain the following conditions (from gauge invariance) for the value of d,

$$\frac{1}{d}\Lambda^2 \to \frac{1}{2}\Lambda^2, \tag{A3}$$

$$\frac{1}{d}\ln\left(\frac{\Lambda^2+m^2}{m^2}\right) \to \frac{1}{4}\left(\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)+\frac{1}{2}\right), \tag{A4}$$

$$\frac{1}{d} \to \frac{1}{4} \quad \text{for finite terms.} \tag{A5}$$

We see that for finite valued integrals when the Wick-rotation is applicable, the condition (10) and the rule (20) gives the usual substitution (A1), but for divergent cases we obtain the identification partially found by [6, 7, 10] and others. Quadratic divergence goes with $d = 2$, logarithmic divergence goes with $d = 4$ plus a finite term (a shift), it is the $+1$ in equation (4). For more than 2 even

[3] The usual method is to calculate the trace (and obtain $d = 4$), but interchanging the order of tracing (multiplication with $g^{\mu\nu}$) and calculating the divergent integrals cannot be proven to be valid.

number of indices generalizations of (A2) should be used, for example in case of 4 indices the substitution

$$l_{E\mu}l_{E\nu}l_{E\rho}l_{E\sigma} \to \frac{1}{d(d+2)} \cdot \left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right) l_E^4. \tag{A6}$$

works.

We emphasize again that for non-divergent integrals the rules (20) and (21) give the same result as the usual calculation (A1).

Appendix B: BASIC INTEGRALS

In this appendix we list the basic divergent integrals calculated by the regularization proposed in this paper. In the following formulae m^2 can be any loop momentum (k) independent expression depending on the Feynman x parameter, external momenta, etc., e.g. $\Delta(x, q, m_a, m_b)$. The regularized integrals are denoted by $\int_{\Lambda\mathrm{reg}}$ meaning $\int_{|k_E|\leqslant\Lambda}$, the integration is understood for Euclidean momenta with absolute value below Λ. The integrals (B1) and (B5) are just given for comparison, those calculated with a simple momentum cutoff.

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{1}{k^2-m^2} = -\frac{1}{(4\pi)^2}\left(\Lambda^2 - m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)\right), \tag{B1}$$

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{k_\mu k_\nu}{(k^2-m^2)^2} = -\frac{1}{(4\pi)^2}\frac{g_{\mu\nu}}{2}\left(\Lambda^2 - m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)\right), \tag{B2}$$

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2-m^2)^3} = -\frac{1}{(4\pi)^2}\frac{g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}}{8}\left(\Lambda^2 - m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)\right), \tag{B3}$$

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{k^2 k_\mu k_\nu}{(k^2-m^2)^3} = -\frac{1}{(4\pi)^2}\frac{g_{\mu\nu}}{4}\left(2\Lambda^2 - 3m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right) + m^2 - \frac{m^4}{\Lambda^2+m^2}\right), \tag{B4}$$

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{1}{(k^2-m^2)^2} = \frac{1}{(4\pi)^2}\left(\ln\left(\frac{\Lambda^2+m^2}{m^2}\right) + \frac{m^2}{\Lambda^2+m^2} - 1\right), \tag{B5}$$

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{k_\mu k_\nu}{(k^2-m^2)^3} = \frac{1}{(4\pi)^2}\frac{g_{\mu\nu}}{4}\left(\ln\left(\frac{\Lambda^2+m^2}{m^2}\right) + \frac{m^2}{\Lambda^2+m^2} - 1\right), \tag{B6}$$

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{k^2 k_\mu k_\nu}{(k^2-m^2)^4} = \frac{1}{(4\pi)^2}\frac{g_{\mu\nu}}{12}\left(3\ln\left(\frac{\Lambda^2+m^2}{m^2}\right) + 5\frac{m^2}{\Lambda^2+m^2} - \frac{m^4}{(\Lambda^2+m^2)^2} - 4\right), \tag{B7}$$

$$\int_{\Lambda\mathrm{reg}} \frac{d^4k}{i(2\pi)^4}\frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2-m^2)^4} = \frac{1}{(4\pi)^2}\frac{g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}}{24}\left(\ln\left(\frac{\Lambda^2+m^2}{m^2}\right) + \frac{m^2}{\Lambda^2+m^2} - 1\right). \tag{B8}$$

(B1)-(B3) depend on the same function of Λ. (B2), (B3) are traced back to (B1) via (20) and (21). (B4) and (B7) have a different Λ dependence. Evaluating these integrals at first step (22) is used, then (20) or (21) can be applied to the remaining free indices.

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