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Chaotic or just complicated? Ball bouncing down the stairs

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Abstract

The aim of this study is to investigate the bouncing dynamics of a small elastic ball on a rectangular stairway and to determine if its dynamics is chaotic. We derive a simple nonlinear recursion for the coordinates of the collisions from which the type of dynamics cannot be predicted. Numerical simulations indicate that stationary bouncing always sets in asymptotically, and is typically quasi-periodic. The dependence on the coefficient of restitution can be very complicated, yet the dynamics is found to be nonchaotic. Only elementary mathematics is required for the calculations, and we offer a piece of user-friendly demo software on our website, http://crnl.hu/stairway, to facilitate further understanding of this complex phenomenon.

Keywords: bouncing dynamics, stair, billiard, gravity, coefficient of restitution, quasi-periodic motion, attractors

(Some figures may appear in colour only in the online journal)

1. Introduction

In an Austrian high school textbook, the bouncing motion of a ball down a stairway is given as an example of chaotic dynamics [1]. Although there is a broad amount of literature on the bouncing dynamics of realistic balls [2–7], as a first attempt, here we study the simplest model of bouncing down the stairs by considering a point-like ball, the bounces are assumed to occur elastically with some energy loss. The investigation of the coefficient of restitution

† Deceased.

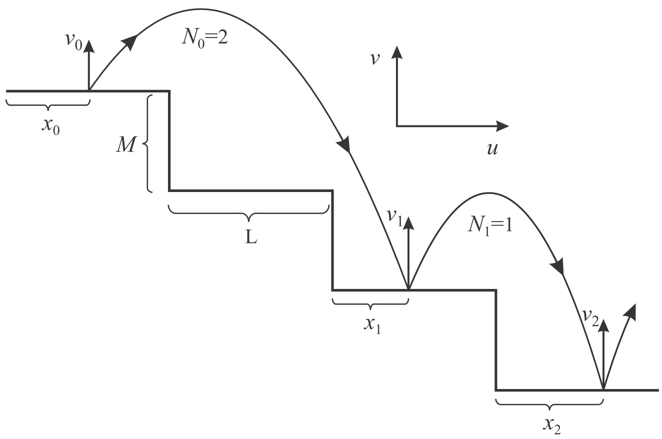

Figure 1. The trajectory of a ball bouncing down a stairway with tread L and rise M, and the quantities characterizing the motion: the location of the nth collision is denoted by x_n. The vertical velocity after the bounce is v_n, and the horizontal velocity is the constant u, while the number of steps the ball jumps over between the nth and $n + 1$st collision is N_n.

(COR) [8–14] and its dependence on different parameters is a current problem of interest. For simplicity, we assume that the COR is constant during the entire bouncing dynamics and takes on a positive value k less than unity.

The traditional direction of study for point-like balls bouncing with a constant COR is the dynamics on a periodically vibrating horizontal plane. This is a paradigmatic example of dissipative chaos, shedding light on aspects of the general validity of certain dynamical systems [15–19]. Bouncing between nonmoving walls defines the broad problem of billiards (see e.g. [20–23]), which are typically considered without any energy loss ($k = 1$), because otherwise with the lack of driving force, any motion would ultimately stop. The literature is much more restricted for billiards in the presence of gravity, some of which, such as wedge-type billiards, are known to support chaotic dynamics [24, 25].

We consider a stairway which consists of horizontal and vertical parts only (see figure 1), and define an *unbounded* billiard problem in the presence of gravity. As in billiards, no friction is assumed between the ball or the stairway, and rotation can be neglected. Collisional energy loss, representing the effect of dissipative processes within the ball via the COR, is natural, and can compensate for the increase in kinetic energy as the height decreases. Thus, a steady state might set in, supporting an ever-lasting translational motion.

An elementary definition of chaos states that chaos is a motion which is ‘irregular in time; unpredictable in the long term, and sensitive to the initial condition; complex, but ordered, in the phase space: it is associated with a fractal structure’ (for more details see the textbook [23]). At first sight, it is hard to decide whether or not the bouncing dynamics is chaotic: the planar, horizontal surface would suggest no chaos (because the surface would behave as a plane mirror not producing any magnification, i.e. a loss of predictability, when illuminated with light). The edges of the steps, however, could be seen as a source of chaos. A detailed investigation of this elementary billiard problem has not yet been carried out. To our knowledge, only the condition for periodic bouncing on every step has been given (see [8], pp 220–222, and [26]). Here we point out that such a motion can only occur with *exceptional* parameters, and the typical behavior is much more involved, but nonchaotic. We offer an exploration of the dynamics with arbitrary COR values as a useful project for undergraduates, supported by a set of problems to solve, and a piece of demo software on the internet.

2. The model

Let us consider a stairway with step tread L and rise M, tilted from left to right (figure 1). Since the coefficient of restitution (COR) is less than 1, the *vertical* component v of the incident velocity of the ball becomes $k < 1$ times smaller during each bounce, while the horizontal component $u > 0$ remains constant.

Our goal is to find a relation between the location and velocity data of the nth and the $n+1$st collision. To keep things simple, we place the origin of our coordinate system on the left-hand edge of the step on which the collision occurs. (This means that we always shift the coordinate x of the collision back to the $(0, L]$ interval.)

Let the coordinate of the nth collision be x_n, and the vertical speed *after* the bounce be v_n. The height of the ball at time t after the bounce, measured from the surface of the step is:

$$y(t) = v_n t - \frac{g}{2}t^2,$$

while the horizontal distance from the origin is $x(t) = x_n + ut$. To determine the time Δt_n that has passed since the last collision, it is worth assuming that we know how many steps are jumped over during this time. Let us call this integer N_n the *jump number*, which will be an important variable in what follows.

For the calculation of Δt_n, we use the fact that at the next collision, that is $n+1$st, the ball will hit the step at height $y = -MN_n$, i.e. $v_n\Delta t_n - (g/2)(\Delta t_n)^2 = -MN_n$, whence

$$\Delta t_n = \frac{\sqrt{v_n^2 + 2gMN_n} + v_n}{g}.$$

The collision occurs with vertical velocity $v_n - g\Delta t_n = -\sqrt{v_n^2 + 2gMN_n}$. The rebound velocity is k times the opposite of this velocity, thus the vertical component after the $n+1$st collision is:

$$v_{n+1} = k\sqrt{v_n^2 + 2gMN_n}. \tag{1}$$

Horizontally, the ball is at a distance of $x_n + u\Delta t_n$ from the origin, on its right. N_n is none other than the number of times the tread L of a step can be included in this interval. Substituting Δt_n,

$$N_n = \left[\frac{x_n + (u/g)(v_n + \sqrt{v_n^2 + 2gMN_n})}{L}\right], \tag{2}$$

where the square brackets represent the integer part. If equation (2) has several solutions, we keep the smallest N_n. So N_n can be calculated from the data of the nth collision and the parameters.

As the origin of the coordinate system is placed on the left-hand end of the step on which the last collision occurs, the coordinate x_{n+1} of the next bounce is the difference of the horizontal displacement and LN_n, that is:

$$x_{n+1} = x_n + \frac{u}{g}(v_n + \sqrt{v_n^2 + 2gMN_n}) - LN_n. \tag{3}$$

System (1)–(3) represents a discrete time dynamics which yields the location and velocity coordinates x_{n+1} and v_{n+1} of the next collision based on the previous ones, x_n, v_n, and the jump number N_n.

3. Dimensionless form

It is worth rewriting the equation of motion in a simpler form, not depending separately, for example, on the tread and rise of the steps, but only the absolute value of the slope $m = M/L$. This can be obtained by dividing (3) by L, yielding the location coordinate measured in units of tread, and by letting velocity v_n appear in units of the constant horizontal velocity, $u > 0$, i.e. via the ratio v_n/u

$$\frac{x_{n+1}}{L} = \frac{x_n}{L} + \frac{u^2}{gL}\left(\frac{v_n}{u} + \sqrt{\left(\frac{v_n}{u}\right)^2 + \frac{2gL}{u^2}mN_n}\right) - N_n. \tag{4}$$

Let us observe that besides m only one parameter, the combination gL/u^2, appears. From here on, this dynamical parameter is referred to as the *length parameter* and is denoted by H. Investigating the other two equations in a similar spirit, we find *no* further parameters.

Therefore, it is useful to rewrite all the equations for the dimensionless speed and location by means of the transformations $v/u \rightarrow v$, $x/L \rightarrow x$:

$$v_{n+1} = k\sqrt{v_n^2 + 2\,mHN_n}, \tag{5}$$

$$N_n = \left[x_n + \frac{1}{H}(v_n + \sqrt{v_n^2 + 2\,mHN_n})\right], \tag{6}$$

$$x_{n+1} = x_n + \frac{1}{H}(v_n + \sqrt{v_n^2 + 2\,mHN_n}) - N_n. \tag{7}$$

It is clear from here that the dynamics depends on three independent parameters:

$$k, \quad m \equiv \frac{M}{L}, \quad H \equiv \frac{Lg}{u^2},$$

the COR, k, the slope, m, and the length parameter, H (while the original set (1)–(3) contains five parameters such as k, M, L, u and g). We shall call the steps long, if the length parameter H is big enough, or more precisely (see problem 1) if $H > 2m$.

Problem 1. In order to interpret the length parameter from another point of view, let us show that a ball launched from the right end of a step with horizontal velocity $u > 0$ will collide with the next step at a location which is $x_i = \sqrt{\frac{2m}{H}}$ times the tread. The steps of a stairway of slope m are considered long if this ratio is smaller than one, namely if $H > 2m$[4].

Generally, steps in buildings are twice as long as their height, hence we use slope $m = 1/2$ in what follows, and the COR will have a wide range of values. As for a classification with respect to the horizontal speed, long steps are natural to examine, on which, for not very energetic motions, several bounces become possible before a jump onto the next step occurs. Therefore we investigate the interval $H \in [2, 8]$. The choice of $H = 4$ is representative for this class, and we shall provide all the numerical results for $H = 4$. In this baseline case $H/(2m) = 4$, meaning that in the case of problem 1, the first collision occurs at the midpoint of the step.

Recursion (5)–(7) is nonlinear, therefore, it might support chaotic dynamics [23, 27]. In the lack of general analytical results, numerical simulations are needed to decide if there exist chaotic solutions in the parameter range investigated. Motivated readers may become better

[4] Detailed solutions of the problems are available on the website http://crnl.hu/stairway.

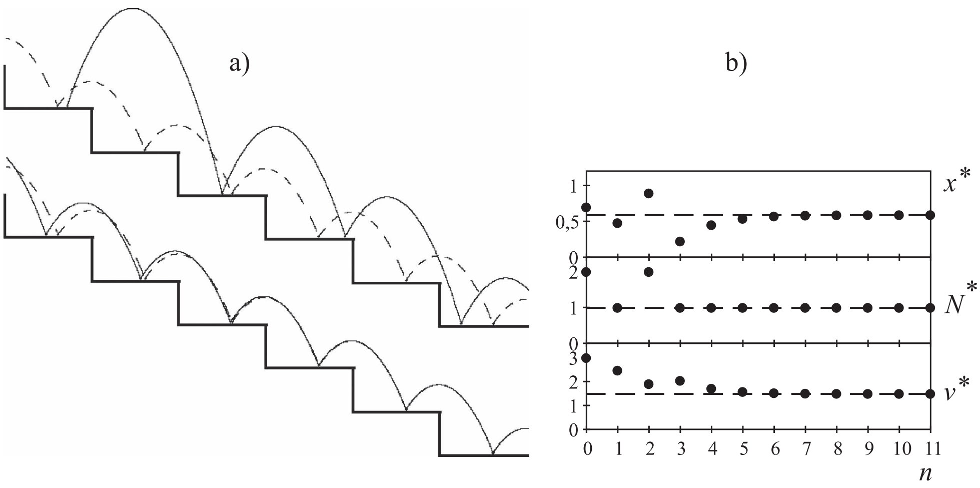

Figure 2. Bouncing with $k = 0.6$ ($m = 0.5$, $H = 4$). (a) A trajectory starting with the initial conditions $x_0 = 0.7$, $v_0 = 3$ (continuous curve, on the lower stairway the motion from the upper part continues) and (b) the time series x_n, v_n, N_n of the bounces. In both panels, the dashed curves represent the attractor, and the dashed horizontal lines indicate the corresponding fixed point values reached after a short transient period. *All* bouncings converge to the attractor of one-step jumps, where $v^* = 1.5$ and $N^* = 1$ because of (8) and (9). The value of x^* depends on the initial condition, here $x^* = 0.592$.

acquainted with the dynamics by running the programs available on the website: crnl.hu/stairway to exhibit the motion of the ball with arbitrary parameters and initial conditions.

4. Simple periodic bouncing (period-one and period-two attractors)

With the typical values of the COR and initial conditions leading to a bouncing motion without any sliding (for more details on sliding, see section 7), one finds that the dynamics *always* reaches a stationary motion, and the ball bounces downwards with a constant average velocity. Over the course of this motion the energy dissipated at the collisions equals the decrease of the potential energy associated with the fall. The restitutional loss is a type of dissipation and, as a consequence, the ball 'forgets' its initial conditions. Therefore, given any set of initial conditions, ultimately the same type of permanent motion is generated. Thus the balls are 'attracted to' a certain dynamical state, which we can term the attractor.

The numerical solution presented in figure 2 is periodic from about the fifth bounce. The simplest attractor occurs here, which is the attractor of periodic bouncing with jump length unity. Panel (b) shows that the series x_n, v_n, N_n converges towards a constant value, to a fixed point.

It is worth mentioning that such periodic bouncing can be observed in the dynamics of realistic balls, too. Because of their final extension, and as a consequence, of the possibility of rotation, and also because of air resistance, our model does not hold for such balls; nevertheless there may be certain qualitative features in common. The demonstration experiment shown in figure 3 illustrates that the attractor of periodic bouncing with the jump length of the tread also exists for realistic balls.

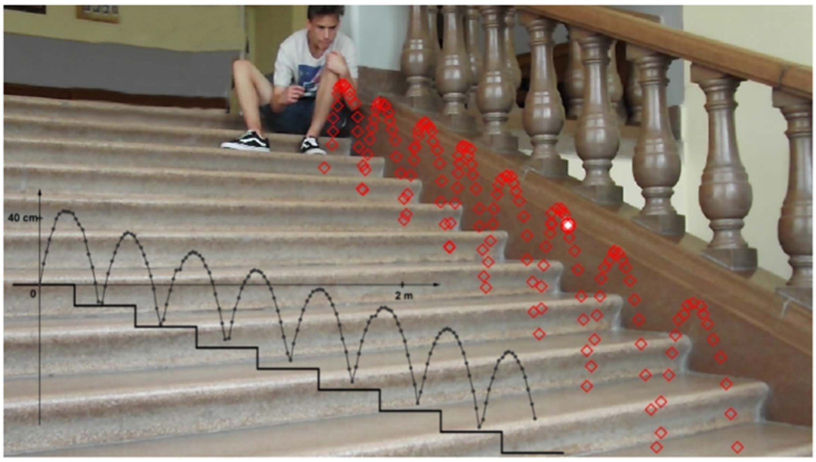

Figure 3. After some experimenting, one is able to find the initial conditions which lead to the periodic motion of a ping-pong ball in the course of which it bounces on each step once. The path of the ball is reconstructed here with red (diamond) symbols from a video recording of the motion by means of the program tracker (http://physlets.org/tracker/). The inset shows the parallax-corrected path. Data: $L = 33$ cm, $M = 13$ cm, $k = 0.8$, $u \approx 0.5$ m s^{-1}.

Returning to our model, from recursion (5)–(7) one can see that the fixed points of the velocity and the jump number are:

$$v^* = 2m\frac{k}{1-k}, \tag{8}$$

$$N^* = \frac{2m}{H}\frac{1+k}{1-k}. \tag{9}$$

Problem 2. Show that results (8) and (9) follow from equations (5)–(7). Of course, one should assume exact repetitions, i.e. $x_n = x_{n+1} = x^*$, $v_n = v_{n+1} = v^*$, and $N_n = N_{n+1} = N^*$.

Special consideration is needed to ensure that N^* should be an integer, by definition. Therefore, it is worth taking an integer value for N^* and searching for the corresponding COR value. It is clear from (9) that only the following *discrete* k values are possible:

$$k_N = \frac{\frac{NH}{2m} - 1}{\frac{NH}{2m} + 1}, \quad N = 1, 2, \ \ldots. \tag{10}$$

This can be called a spectrum of CORs, since periodic bouncing can only occur at discrete k values, in a similar way to the discrete energy levels of the hydrogen atom [28–30], as indicated graphically in figure 4. For $H = 4$ (the baseline case), $k_1 = 3/5 = 0.6$ for the periodic attractor jumping over one step, and $k_2 = 7/9 = 0.77$ for the periodic motion that jumps over two steps.

Note that for length parameter $H = 1$ the first element k_1 of the spectrum (as well as the corresponding v^*) is zero: the simplest periodic bouncing with $N = 1$ cannot occur. For the range $H \in [2, 8]$ of the length parameters investigated, the k_1 value falls into the interval

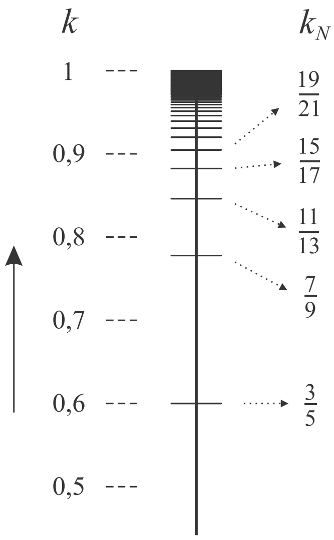

Figure 4. The k_N spectrum of periodic bouncing over N steps. An increasing N results in a more and more dense set for increasing k_N values.

(1/3, 7/9), i.e. in an intermediate and realistic range of CORs. This allows us to conveniently investigate the phenomena characterizing both the $k < k_1$ and the $k > k_1$ regions, since neither of them is too short.

Problem 3. Based on (10), what is the H parameter belonging to a periodic bouncing over N steps?

Problem 4. What is the fixed point value v^* of the vertical velocity on the N^{th} level of the spectrum?

Problem 5. Figure 4 indicates that the spectrum gets more and more dense at high N values (just like the spectrum of the hydrogen atom), as the values for k approach 1. Let us show that in this region, the following holds with good accuracy:

$$k_N = 1 - \frac{4m}{H}\frac{1}{N}, \quad N \gg 1. \tag{11}$$

Remark: The caption to figure 2 states that x^* depends on the initial condition; this is true for all k_N values. The reason becomes clear when one studies figure 2(a) and notices that the important characteristic of the attractor is the parabola arch of the bouncing motion. A given value of N^* and v^* determines a given parabola arch; x^* may not be unique, however. From the 'point of view' of the ball, the floor under its attractor curve can be shifted to the left or right, since it will still be bouncing on horizontal surfaces with the same height difference. Hence different x^* values might belong to the attractor determined by N^* and v^*.

It may happen that in a stationary motion, bouncing is repeated only after every second collision. This means that before the first and the second collision, the ball jumps over N and K steps, respectively (N, K are positive integer numbers), and this pattern is repeated ad infinitum.

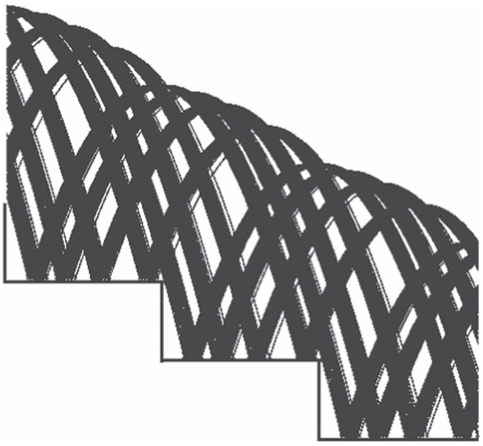

Figure 5. The motion with $k = 0.75$ after the transients have died out (other parameters are the same as in figure 2). Here, the trajectories are plotted between 500 and 1500 bounces in this representation of the attractor: after the third collision, the trajectories enter the picture at the same height over step 0, where they previously left the right-hand edge of the picture over step 3, again and again. Even when starting from different initial conditions, the motion ends up conforming to this quasi-periodic dynamics. For this k, $k_{1,2} < k < k_2$, i.e. the COR is in between the values belonging to the $N = 1$, $K = 2$ two-cycle and the $N = 2$ one-cycle. As a consequence, during the quasi-periodic motion, the ball jumps over one or two steps at once, the latter being more frequent since the long-term average of the jump number is found to be $\bar{N} = 1.747$ numerically. The corresponding v_n velocity time series is shown in the inset of the next figure.

It is also worth noting that these two-cycles exist only for *exceptional* k values determined by N and K. The COR values $k_{N,K}$ associated with these numbers are easy to determine. Among them, the smallest one is $k_{1,2}$ (for $H = 4$ it takes the value $k_{1,2} = 0, 715$) which lies between the k_1 and k_2 COR of the simplest periodic bouncings.

Problem 6. Derive from (5)–(7) the 'eigenvalue equation' which determines the $k_{N,K}$ COR spectrum of two-cycles.

5. Motion with arbitrary *k* values

With arbitrary COR values when k is not equal to the ones belonging to one- or two-cycles, a long-term, quasi-periodic motion is found to set in, as figure 5 illustrates. The terminology comes from the property of the motion not repeating itself exactly: it is only similar to periodic motion (for a more detailed definition see [23, 27, 31]), and such dynamics are known to be nonchaotic.

To understand the essence of this phenomenon, let us imagine that we begin with COR k_1. In this case, a simple periodic motion is generated: the ball bounces on each step, at the same location and with the same velocity. By increasing the value of k somewhat, the jumps become longer because of the smaller energy loss. Among the $N = 1$ jumps, there will be some $N = 2$ jumps as well. By further increasing k, the number of $N = 2$ jumps increases monotonically compared to the ones with $N = 1$, and in the end only $N = 2$ remains. Here we arrive at the case of k_2. There is an intermediate value (but not the arithmetic mean of k_1 and k_2) where the number of jumps with $N = 1$ and $N = 2$ are equal, and they alternate with one

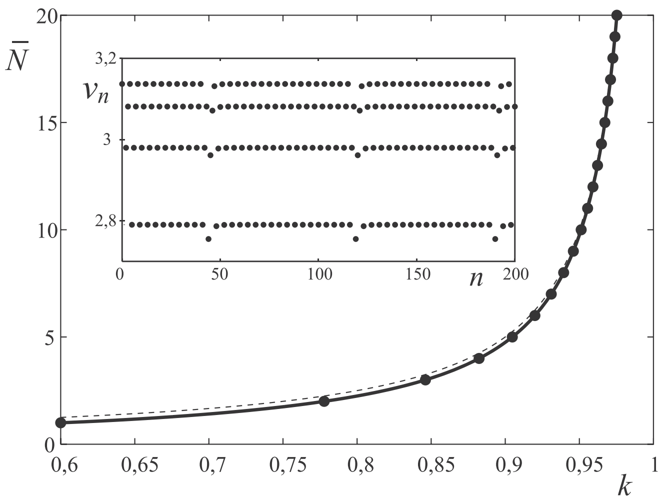

Figure 6. The average jump number $\bar{N}$ on the attractor versus k, based on numerical simulations for large CORs: $k \geqslant k_1$ ($m = 0.5$, $H = 4$). It can be clearly seen that $\bar{N}$ increases monotonically with increasing k. Discrete dots mark the points k_N, N, i.e. the one-cycles with jump number N. The dashed line stands for the approximate relation $\bar{N} = (2(1-k))^{-1}$, which is valid for large k (and this seems to be a rather good approximation for the whole range). Naturally, for the $N, N+1$ two-cycles $\bar{N}(k_{N,N+1}) = (N + (N+1))/2 = N + 1/2$, and these two-cycles also correspond to certain points of the continuous curve, but we do not indicate them for better visibility. The inset shows the velocity time series v_n of the $k = 0.75$ quasi-periodic attractor of the previous figure. It is clearly seen that the motion is almost repeated after every fourth bounce, but the 'outliers' always prevent exact repetition.

another. The quasi-periodic attractor is converted here into a two-cycle, with the corresponding COR of $k_{1,2}$.

These observations and the numerical evidence show that for two-cycle attractors only $K = N + 1$ is possible, in other words, the longer jumps of the two-cycles must not exceed the length of the shorter one by two or more steps. Numerical evidence also indicates that no cycle of length 3 or longer exists in the $k \geqslant k_1$ range of the CORs. For every k in $k_N < k < k_{N+1}$ (where $N \geqslant 1$), the quasi-periodic motion is typical, and composed of jumps of length N and $N + 1$. As k increases, so does the number of $N + 1$ jumps compared to those of N.

To characterize the process, it is useful to determine the average jump number $\bar{N}$ on the attractor, giving the number of steps that are jumped over by the ball on average between two subsequent collisions. Figure 6 shows the smooth, monotonic increase of $\bar{N}(k)$. Note that for CORs belonging to one-cycles of jump N, this N is also the average jump number: $\bar{N}(k_N) = N$. For $N \gg 1$, when the k_N are close to 1, the COR values become dense and according to the inverse of (11), $\bar{N} = 4\,m/H \cdot 1/(1-k)$. The average jump number thus increases proportionally to $(1-k)^{-1}$.

6. Multiple bounces on a single step

In the region of small CORs, for $k < k_1$ new forms of motion appear as well. When looking for two-cycles, N_n was never 0. For small CORs, however, this makes sense and indicates two

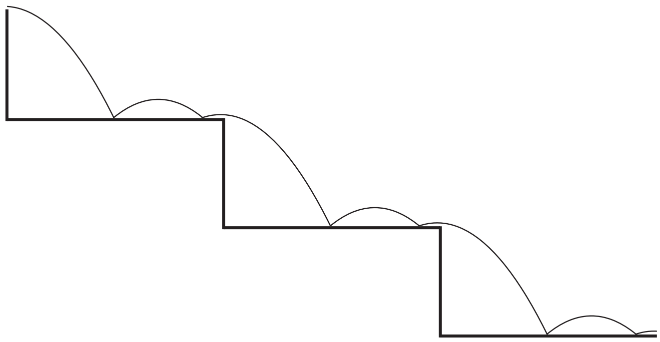

Figure 7. A double bounce on a single step. The attractor trajectory with $k_{1,0} = 0.405$, after the dying away of the transients (all the other parameters as in figure 2). The bounce with any initial condition converges to a periodic attractor, a two-cycle, where the ball bounces twice before moving onto the following step.

possible bounces on the same step. The two-cycle for which the ball jumps to the following step, bounces on it twice, and this repeats (figure 7), is characterized by the indexes $N = 1$, $K = 0$ or $N = 0$, $K = 1$. The corresponding COR value $k_{1,0} = k_{0,1}$ is found to be $k_{1,0} = 0.405$ for $H = 4$, and is well below k_1. Since in this case the ball moves to the next step after only two bounces, the average jump number is $1/2$: $\bar{N}(k_{1,0}) = 1/2$.

Problem 7. Derive the expression of $k_{1,0}$ for arbitrary parameters.

For even smaller COR values, the ball may bounce three times before jumping to the following step, where everything is repeated. When the ball bounces j times on a step (j is an arbitrary natural number) the motion is a $(j + 1)$-cycle repeating itself after $j + 1$ bounces. The average step number is then $1/(j + 1)$. The corresponding COR value (which decreases with an increasing j) can be determined using the method previously indicated (see problem 11).

7. Sliding

In the case of sufficiently small CORs (strong dissipative losses), the ball, after a period of bounces on different steps, might undergo an *infinite* number of collisions on a single step. At the end of this process, the ball remains on the surface of the step, and as its horizontal speed is constant, the subsequent motion should be interpreted as sliding in continuous time. Iterations (5)–(7), characterizing the bouncing dynamics, ought to be augmented if one is interested in the sliding sequence. (In the language of discrete time dynamics (5)–(7) the ball appears to stop, since $v_n \to 0$ and there is no displacement between the bounces for $n \to \infty$; the real and 'iterational' times become completely different, and the former appears to stop in the latter[5].) For simplicity's sake, we do not analyze the details of the sliding, and stop

[5] It follows from equation (3) that for bounces on the same step, i.e. with $N_n = 0$, $x_{n+1} - x_n = u2v_n/g$. For a large number of collisions on the same step $v_n \to 0$, and the horizontal displacement $\Delta x = x_{n+1} - x_n$ between subsequent bounces approaches zero, i.e. the ball seems to stop in discrete time. Since the time $\Delta t = 2v_n/g$ between two bounces also goes to zero, we nevertheless recover that the horizontal speed $\Delta x/\Delta t = u$, is constant (this constant is unity in dimensionless variables).

following the iterations after infinite bounces occur on a given step, considering the dynamics to have reached the *attractor* of sliding.

Let the vertical velocity right after the first bounce on a given step be denoted by v_i; the total horizontal displacement after infinitely many collisions is then

$$\Delta x = \frac{2v_i}{H}\frac{1}{1-k}. \tag{12}$$

Problem 8. Derive relation (12).

If the first impact on a given step occurs at location x_i, the condition for the set-up of sliding is that the ball is still before the endpoint of the step of coordinate unity after an infinite number of collisions, i.e. $x_i + \Delta x < 1$. Substituting relation (12), after rearrangement

$$v_i < \frac{H}{2}(1-k)(1-x_i). \tag{13}$$

Whether this inequality is fulfilled depends, with a given k, on the impact coordinate and velocity, x_i and v_i, respectively. We consider our simulation to have reached the attractor of sliding if inequality (13) holds for x_i and v_i upon the impact of a given step.

The critical COR value k_c below which *any* initial condition leads, after transients, to sliding can be determined based on the numerical evidence as follows. As the horizontal velocity is constant, the ball leaves the previous step with a (dimensionless) horizontal velocity of unity, and the arch of the oblique projection crosses the surface of the next step at an x_i value with a rebound velocity v_i , with which the series of an infinite number of collisions stops exactly at the edge of the step, i.e. inequality (13) is converted into an equality. Thus, we obtain

$$\sqrt{\frac{2m}{H}}\frac{1+k_c}{1-k_c} = 1. \tag{14}$$

Problem 9. Derive relation (14).

Rearranging (14), the explicit result for k_c is

$$k_c = \frac{\sqrt{\frac{H}{2m}} - 1}{\sqrt{\frac{H}{2m}} + 1}. \tag{15}$$

For $H = 4$, this yields $k_c = 1/3 = 0, 33$. For CORs smaller than k_c, it is found to be true that all motions end in sliding, i.e. long-term bouncing dynamics does not exist.

8. Motion with small *k* values

Decreasing k somewhat below k_1 (the value belonging to the one-step periodic bouncing), we still find that only a *single* asymptotic attractor exists: the bouncing motions[6] from arbitrary

[6] The previous arguments allow us to give a precise definition of the bouncing motion as dynamics for which iteration (5)–(7) leads to *nonzero* velocities v_n for $n \to \infty$.

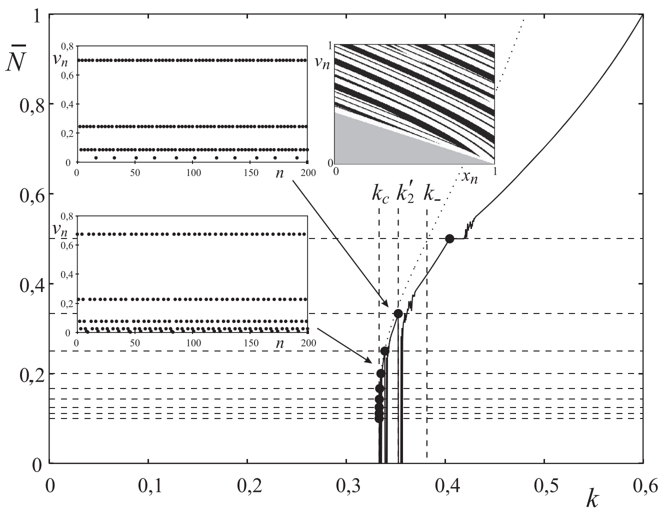

Figure 8. The average jump number $\bar{N}$ on the attractor of bouncing motion versus k in the range of small CORs ($k \leqslant k_1$), obtained numerically ($m = 1/2$, $H = 4$). Horizontal dashed lines mark the values $\bar{N} = 1/2, 1/3, \ldots 1/10$, black dots represent the COR values k_j' belonging to period-($j + 1$) attractors. The value of k_- is marked with a vertical dashed line. The dotted curve denotes the approximate form of $\bar{N}(k)$ valid close to k_c. The two insets on the left exhibit the quasi-periodic time series v_n for a COR value $k = 0.35$ (somewhat below $k_2' = 0.353$) and $k = 0.337$. On the right-hand side of the upper panel, the basins of attractors of the coexisting bouncing and sliding motions at $k = 0.35$ can be seen. Arrows point to the $\bar{N} = 0,298$ and $\bar{N} = 0,194$ values belonging to the insets.

initial conditions become qualitatively similar asymptotically. The attractor is typically quasi-periodic and the numerically determined average jump number $\bar{N}$ decreases with k (see figure 8). There exists, however, a COR value k_-, below which this does not hold, because for certain initial conditions sliding sets in, and the quasi-periodic attractor of bouncing and the attractor of sliding *coexist*. For $H = 4$, numerics indicates that $k_- = 0,382$. In such cases, a question arises regarding their basins of attraction. The basins can be determined by coloring the points of the plane x_0, v_0 of initial conditions according to the attractor to which the motion emanating from them tends. The right-hand upper inset of figure 8 exhibits the basins of attraction of bouncing and sliding in white and black, respectively, for $k = 0.35$ and $H = 4$. The gray triangle marks the region in which inequality (13) holds, and the initial condition from here can only lead to sliding[7]. The smoothness of the boundaries indicates that not even the transient form of chaos is present in the dynamics, since the fractal basin boundaries are the consequences of nonattracting chaotic sets [23, 27, 32].

[7] The gray triangle exists for any COR value and represents the initial conditions leading to sliding on the very first step, which are therefore not of interest from the point of view of our investigation of bouncing motion. The basic difference between the ranges $k > k_-$ and $k < k_-$ is that in the latter, permanent bouncing might coexist with motion which is converted into sliding *after* a finite number of bounces on different steps. In the case of such coexistence, the average jump number, $\bar{N}$, in figure 8 refers to that of the attractor of bouncing motion. When all initial conditions lead to sliding, we take $\bar{N} = 0$, since this quantity yields the typical jump number between subsequent bounces; but in such cases the total distance is less than 1, even after infinite bounces.

Problem 10. Estimate the value of k_- based on the fact that inequality (13) holds not only for the initial conditions, but also for typical values on the attractor of the bouncing motion, indicating that the dynamics might enter the region of the gray triangle. Hint: use the observation that (8) provides a good approximation for the average velocity for any long-term bouncing, and can thus be taken to represent typical values of v_i, and $1/2$ can be considered to be the typical value of x_i.

Below k_- (but above k_c), the change of the average jump number is not monotonic, but the trend of $\bar{N}$ is a decreasing one. It is surprising that long before reaching k_c from above, the option of a bouncing motion disappears in short COR intervals, and $\bar{N} = 0$. When k decreases within such an interval, its end is designated by the sudden appearance of a finite $\bar{N}$, which is the local maximum. These $\bar{N}$ values are the reciprocals of the integers and represent particular periodic long-term bouncing motions: after a bounce with $N = 1$, the ball collides with the same step j times. The corresponding COR values are denoted by k_j' and the average jump number is, of course, $\bar{N}(k_j') = 1/(j + 1)$. The longest such interval exists in between $k_2' = 0.3527$ and $k = 0.3555$, and the others repeat themselves with ever shorter lengths when approaching k_c from above. This accumulation can clearly be seen in figure 8.

Problem 11. Determine the COR values k_j' based on the fact that the periodic time series of N_n belonging to them can be chosen as $N_0 = N_1 = \cdots = N_j = 0$, $N_{j+1} = 1$, which is repeated ad infinitum. Hint: use recursions (5)–(7), which become particularly simple for $N_n = 0$.

Problem 12. Determine the form of function $\bar{N}(k)$ in the vicinity of k_c based on the accumulation of the COR values k_j' in this region, i.e. for large j.

For $k < k_c$ only the attractor of sliding exists. Reaching the value of k_c from below, bouncing motions appear 'suddenly': the curve $\bar{N}(k)$ starts with an infinite slope. The appearance of bouncing motion at k_c resembles the change characterizing the phase transitions of the second order or, in the language of dynamical systems, bifurcations [27, 31].

9. Summary

The aim of our study has been to investigate whether the bouncing motion of a ball down a stairway is chaotic—at least in the simplest model of this phenomenon. Interestingly, complex behavior has only been found in the regime of small CORs, i.e. strong dissipation. The recursive dynamics (5)–(7) is nonlinear, designated by the opportunity for coexisting attractors, and the bifurcation occurring at k_c. The basin boundaries are, however, smooth, as the inset of figure 8 illustrates. The function $\bar{N}(k)$ for $k < k_1$ is not smooth: it exhibits sudden jumps at discrete points. We investigated how the difference of any of the coordinates between a pair of long lasting bouncing motions changes in time, starting from nearly identical initial conditions. Instead of a rapid (exponential) increase, we observed a *decrease* on all occasions: the average Lyapunov exponent [23, 27] is found to be negative for any k in the investigated parameter range. We thus conclude that the dynamics of the investigated model is *not* chaotic. The dynamics is nevertheless complex, which is reflected by the fact that analytical forms are not found for many quantities, among others for the average jump number $\bar{N}$, and numerical simulations are unavoidable. This complexity suggests that the dynamics might turn chaotic after minor modifications to the shape of the step. In fact, a rounded transition from a horizontal surface to a vertical one at the edge of the steps would turn the billiard into a scattering (convex) billiard. Thus, in the case of a sufficiently large radius of curvature, one expects the appearance of robust chaos in the smoothed-out billiard problem.

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References

[1] Jaros A, Nussbaumer A and Kunze H 1999 *Basiswissen Physik-Compact* (Vienna: Öbv&hpt)
[2] Cross R 2002 Grip-slip behavior of a bouncing ball *Am. J. Phys.* **70** 1093–102
[3] Cross R and Nathan A M 2007 Experimental study of the gear effect in ball collisions *Am. J. Phys.* **75** 658–64
[4] Wadhwa A 2013 Study of the dynamic properties and effects of temperature using a spring model for the bouncing ball *Eur. J. Phys.* **34** 703
[5] Cross R 2015 Behaviour of a bouncing ball *Phys. Educ.* **50** 335
[6] Chastaing J-Y, Bertin E and Géminard J-C 2015 Dynamics of a bouncing ball *Am. J. Phys.* **83** 518–24
[7] Cross R 2010 Enhancing the bounce of a ball *Phys. Teach.* **48** 450–2
[8] Stronge W J 2000 *Impact Mechanics* (Cambridge: Cambridge U. P.)
[9] Cross R 2008 Cue and ball deflection (or squirt) in billiards *Am. J. Phys.* **76** 205–12
[10] Cross R 2000 The coefficient of restitution for collisions of happy balls, unhappy balls and tennis balls *Am. J. Phys.* **68** 1025–31
[11] Thornton S T and Marion J B 2003 *Classical Dynamics of Particles and Systems* (Pacific Grove, CA: Brooks/Cole)
[12] Müller P, Heckel M, Sack A and Pöschel T 2013 Complex velocity dependence of the coefficient of restitution of a bouncing ball *Phys. Rev. Lett.* **110** 254301
[13] Aguiar C E and Laudares F 2003 Listening to the coefficient of restitution and the gravitational acceleration of a bouncing ball *Am. J. Phys.* **71** 499–501
[14] Heckel M, Glielmo A, Gunkelmann N and Pöschel T 2016 Can we obtain the coefficient of restitution from the sound of a bouncing ball? *Phys. Rev.* E **93** 032901
[15] Pieranski P 1983 Jumping particle model. Period doubling cascade in an experimental system *J. Phys. (Paris)* **44** 573
[16] Franaszek M and Pieranski P 1985 Jumping particle model. Critical slowing down near the bifurcation points *Can. J. Phys.* **63** 488
[17] Kowalik Z J, Franaszek M and Pieranski P 1988 Self-reanimating chaos in the bouncing-ball system *Phys. Rev.* A **37** 4016–22
[18] Luck J M and Mehta A 1993 Bouncing ball with a finite restitution: chattering, locking, and chaos *Phys. Rev.* E **48** 3988–97
[19] Barroso J J, Carneiro M V and Macau E E N 2009 Bouncing ball problem: Stability of the periodic modes *Phys. Rev.* E **79** 026206
[20] Berry M V 1978 Regular and irregular motion *Topics in nonlinear mechanics Am. Inst. Ph. Conf. Proc.* 46, 16–20
[21] Berry M V 1981 Regularity and chaos in classical mechanics, illustrated by three deformations of a circular billiard *Eur. J. Phys.* **2** 91–102
[22] Bunimovich L A 1991 Conditions of stochasticity of two-dimensional billiards *Chaos* **1** 187–93
[23] Tél T and Gruiz M 2006 *Chaotic Dynamics* (Cambridge: Cambridge University Press)

[24] Korsch H J and Jodl H-J 1999 *Chaos, A Program Collection for the PC* II Ed (Berlin: Springer)
[25] Hartl A E, Miller B N and Mazzoleni A P 2013 Dynamics of a dissipative, inelastic gravitational billiard *Phys. Rev.* E **87** 032901
[26] Aston P J and Shaif R 2007 The dynamics of a bouncing superball with spin *Dynamical Systems* **22** 291–322
[27] Ott E 1993 *Chaos in Dynamical systems* (Cambridge: Cambridge University Press)
[28] French A P and Taylor E F 1978 *An Introduction to Quantum Physics* (New York: Norton)
[29] Saxon D S 2012 *Elementary Quantum Mechanics* (Mineola, NY: Dover Publications)
[30] Griffiths D J 2004 *Introduction to Quantum Mechanics* II Ed (Upper Saddle River, NJ: Prentice Hall)
[31] Strogatz S 2001 *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (Boulder: Westview Press)
[32] Lai Y-C and Tél T 2011 *Transient Chaos* (New York: Springer)