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On the importance of the convergence to climate attractors

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Abstract. Ensemble approaches are becoming widely used in climate research. In contrast to weather forecast, however, in the climatic context one is interested in long-time properties, those arising on the scale of several decades. The well-known strong internal variability of the climate system implies the existence of a related dynamical attractor with chaotic properties. Under the condition of climate change this should be a snapshot attractor, naturally arising in an ensemble-based framework. Although ensemble averages can be evaluated at any instant of time, results obtained during the process of convergence of the ensemble towards the attractor are not relevant from the point of view of climate. In simulations, therefore, attention should be paid to whether the convergence to the attractor has taken place. We point out that this convergence is of exponential character, therefore, in a finite amount of time after initialization relevant results can be obtained. The role of the time scale separation due to the presence of the deep ocean is discussed from the point of view of ensemble simulations.

1 General considerations

A recently emerging view in climate science claims that the relevant quantities from a climatic point of view are the statistics taken over an ensemble of possible realizations evolved from various initial conditions; see e.g., [1–5] for low-order models, and [6–12] for general circulation models (GCMs). In this paper we argue that investigating whether the ensemble has converged to a dynamical attractor is important. Such an investigation is lacking from certain large scale simulations (see [6–10]). We claim that large scale models, too, should be augmented by a careful study of the convergence.

The relevance of attractors for the climate system stems from its unpredictability. In this context unpredictability means that even if the initial condition, corresponding to a given time instant, is approximately known, the system evolves in considerably different ways from slightly differing initial conditions that comply with the approximate knowledge. In other words, the dynamics is chaotic-like. Right after

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initialization, the possible ways of evolution are similar to each other and are strongly determined by the approximate initial condition. Weather forecasting concentrates on this period. In particular, probabilistic weather prediction aims to uncover the possible ways of evolution [13] on the short time scale of days or weeks. With time passing, however, the set of the possible ways of evolution is, due to the nonlinearity of the equations of motion, broadening. We argue that a long-term "final" plethora of these ways of evolution is what is of main interest for climate research.

Asymptotically, the set of the possible ways of evolution *converges* to a so-called dynamical attractor, and its distribution on the attractor (which is not uniform) also becomes unique. Uniqueness means *independence* of the initial conditions, including independence of the above-mentioned approximate knowledge. This distribution is, in mathematical terms, the natural probability distribution of the dynamical attractor of the climate system. It is clear that the natural probability distribution of the attractor is what defines the probabilities of all possible weather situations and, more generally, of all states of the whole system that are permitted by its dynamics. On the one hand, being independent of the initial state, this is the probability distribution that faithfully reflects what is called the internal variability of the climate system. In particular, the width (the higher order moments) of the probability distribution characterizes the strength of the internal variability. On the other hand, the expectation values of the physical quantities provide the climatic mean values. A novel feature of the approach is that it enables one to obtain both characteristics, i.e., the internal variability and the climatic mean values, simultaneously. In fact, all this implies that the climate itself is best defined as the attractor along with its natural distribution. If, however, expectation values are evaluated too early, they do not characterize the climate, or the attractor, but rather they characterize how the convergence towards the attractor takes place, also reflecting properties of the initialization. It is this subjective character of ensemble simulations that can be excluded by waiting long enough, until the convergence to the attractor takes place, after which the ensemble results reflect the objective properties of the climate.

In more technical terms, in the language of dynamical systems theory, in the unrealistic case when a stationary forcing is imposed on the climate system, its dynamical attractor is a usual chaotic attractor [14] and is also stationary in time. If the forcing is of general time-dependence, as in the climate dynamics during climate change, then the attractor is called a snapshot [15] or pullback [1,16] attractor (for experimental realizations, see [17,18]) which is well-defined in any particular time *instant* and also has its own *time evolution*. This implies the time evolution of the climatic means and, more generally, of the relevant probability distribution, which can be interpreted as a *climate change* [3]. Since this time evolution is uniquely determined by the forcing scenario the climate system is subject to, this evolution represents what is called [10,25] the "forced response" of the climate system [11].

Our studies [19,20] indicate that the process of convergence of an ensemble to the attractor and its natural probability distribution from any set of initial conditions is "fast": it is expected to have a characteristic time scale and to be "exponential-like", i.e., faster than any power-law. This is a consequence of the dissipative nature of the dynamics. It is due to the exponential character that the convergence possesses a characteristic time, τ,[1] and we can say that after some t^*, which is a few times multiple of τ added after the time of initialization t_0 (i.e., $t^* = t_0 + t_\mathrm{c}$ with $t_\mathrm{c} \propto \tau$), a convergence to the attractor and its natural distribution has taken place (with an exponential accuracy).[2]

[1] In autonomous dissipative mixing dynamical systems τ is understood formally as the reciprocal of the first nonzero eigenvalue of the transfer operator [21,22].

[2] In fact, one can associate distinct t^* values to distinct observables. Therefore, a more specific recommendation of ours is that the convergence should be checked separately for

It is from t^* on that the ensemble characterizes the full plethora of possibilities, along with their appropriate probabilities, that is permitted by the climate dynamics. It is thus only after t^* that one can clearly distinguish the forced response, as the shift in climatic mean values, from the effect of internal variability [19,20]. Furthermore, the standard deviations over the ensemble provide a measure for the strength of internal variability only then, as well. After t^*, any element of the ensemble represents a possible evolution of the climate system. Some authors consider a description based on ensembles that have converged to attractors as the "*theory of parallel climate realizations*", as discussed in [23]. We emphasize that evaluating any statistics earlier than t^* leads to results that do depend on the initial condition (like in weather forecast), and are thus, strictly speaking, not relevant from the point of view of the climate and its change.

2 Investigating the convergence

The process of converging to the attractor should be investigated numerically by taking *different sets* of initial conditions and following the time evolution of the corresponding ensembles. In practice, one may initialize one ensemble relatively far in the past before the time interval of interest, and investigate how another ensemble, initialized later, converges to the previously initialized one. Similarly in part to how it was done in [19] for a low-order model, we shall illustrate the convergence via numerical examples in the intermediate-complexity GCM, the Planet Simulator [24], in a version treating the upper ocean as a heat reservoir with prescribed heat fluxes. The details of our model setup are discussed in [20,23]. In particular, we use the default settings except for the depth of the mixed-layer ocean which we take to be 200 m.

The lower graph in Figure 1 presents the forcing scenario, prescribed via the atmospheric CO_2 concentration. We initialize our first ensemble, consisting of 40 members, at $t_0 = 0$. Initially, the atmosphere is at rest, and the difference between the ensemble members is obtained by randomly perturbing the surface pressure field by an amount on the order of 10 hPa. The ensemble average of the annual mean surface temperature of Earth as a function of time is plotted in Figure 1 as a gray line turning into black at $t^* = 200\,\mathrm{yr}$. This line covers the entire time span of 1500 yr. After an initial transient lasting up to $t^* = t_0 + t_c \approx 200\,\mathrm{yr}$, the average is constant, $T_{\mathrm{cold}} \approx 8°\mathrm{C}$, up to $t = 600\,\mathrm{yr}$, and this already indicates that a convergence has taken place to the attractor corresponding to the initially constant CO_2 concentration of 360 ppm (this is why we change gray into black at t^*).

Since in the period $0 < t < t^*$ the climate is the same cold stationary climate of 360 ppm, just as for $t^* < t < 600\,\mathrm{yr}$, the mean climatic temperature is T_{cold} also for $0 < t < t^*$. This value we mark by a black horizontal line segment in the period $0 < t < t^*$, too, in order to indicate that the attractors exists, and carries the same average as later also in this interval.[3] The deviation of the average temperature T taken with respect to our ensemble (gray line) for $0 < t < t^*$ from T_{cold} (black) illustrates our message: the ensemble average (gray) is well-defined but has no climatic relevance. Conclusions drawn from the ensemble before convergence to the attractor takes place

various different observables of particular interest as part of a particular climate-oriented investigation.

[3] Note that this is so only because the forcing is constant. Otherwise, such an extrapolation of the attractor average is not possible, so that the attractor average is *not* available before the convergence of the first ensemble. It can, however, always be generated by another ensemble initiated much earlier (at negative times in our example). Numerically, the constant value plotted for $[0, 200]$ yr is obtained here as the temporal average of the black line for $[200, 600]$ yr.

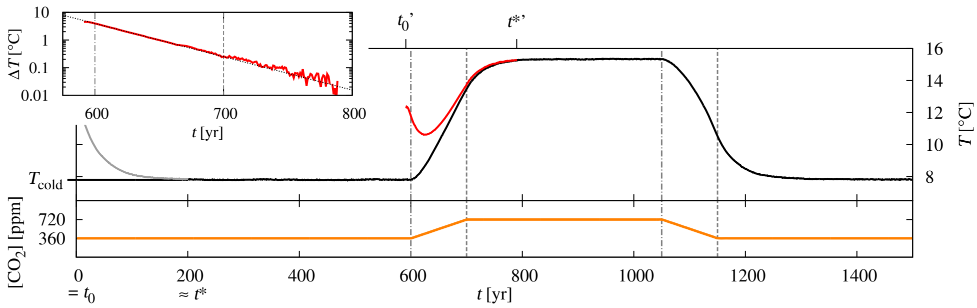

Fig. 1. The annual global surface mean temperature T as a function of time in different ensembles. The ensemble average taken over the ensemble initialized at $t_0 = 0$ is shown in gray from 0 to 200 yr and in black from 200 yr on when it represents the average on the attractor. The constant black line before 200 yr represents the average on the attractor at these time instants (T_{cold}; see text, and also Footnote 3). The entire black line represents the time evolution of the average on the snapshot attractor for all t. The red line from 591 yr to 790 yr [also gray in print] marks the ensemble average taken over the ensemble initialized at $t_0' = 591$ yr, before the average merges at $t^{*\prime}$ with the black line designating the snapshot attractor. In the inset, we show the difference ΔT of the red and the black lines as a function of time. We also include here an exponential fit (marked dotted) for the interval $[596, 656]$ yr which yields a relaxation time $\tau = 36$ yr. In the main plot, the CO_2 concentration, i.e., the forcing, is also given (in orange [gray in print], see the bottom graph), as well as t_0 and t_0' (the time instants of initialization), and t^* and $t^{*\prime}$ (the time instants up to which the convergence of the ensemble averages to the attractor values takes place with an accuracy comparable to the size of the numerical fluctuations). The vertical dot-dashed (dashed) lines in gray mark the beginning (end) of the linear ramps in the CO_2 concentration in the entire figure.

may lead to temperatures strongly different ($2\dots3\,°$C higher) from the climatic mean T_{cold}. It is only for $t > t^*$ where our ensemble properly characterizes the climate (after gray and black merge).

The average on the attractor (black line) starts changing at $t = 600$ yr, and is seen to roughly follow the later linear increase and decrease (from $t = 600$ yr and $t = 1050$ yr, respectively) in the CO_2 concentration[4]. In this period the driving is not constant, the attractor underlying the climate thus cannot be a traditional attractor, but it is a snapshot attractor. It turns out to be strongly time-dependent in this period, indicating *climate change*. We emphasize that this attractor (represented by the black line) reflects the dynamical attractor of the problem in the whole time span investigated.

We initialize a second ensemble (also of 40 members) at $t_0' = 591$ yr with an algorithm similar to that of the first one. The average taken over the second ensemble, marked by red [gray in print] in Figure 1, converges to the black line, i.e., to the average on the snapshot attractor (this is numerically generated, by this time, as the ensemble average over the first ensemble), within about $t_c' = 200$ yr, i.e., by $t^{*\prime} = t_0' + t_c' \approx 790$ yr. The convergence times t_c and t_c' to the attractor turn thus out to be approximately the same for t_0 and t_0'. According to the inset of Figure 1, the difference ΔT between the red [gray in print] and the black lines of the main plot decreases in time exponentially, as $\exp(-t/\tau)$, with a relaxation time of $\tau = 36$ yr with

[4] It is worth observing that the temperature reaches both its upper and lower plateau with a time delay (of about 100 yr) compared to the beginning of the corresponding CO_2 plateaus.

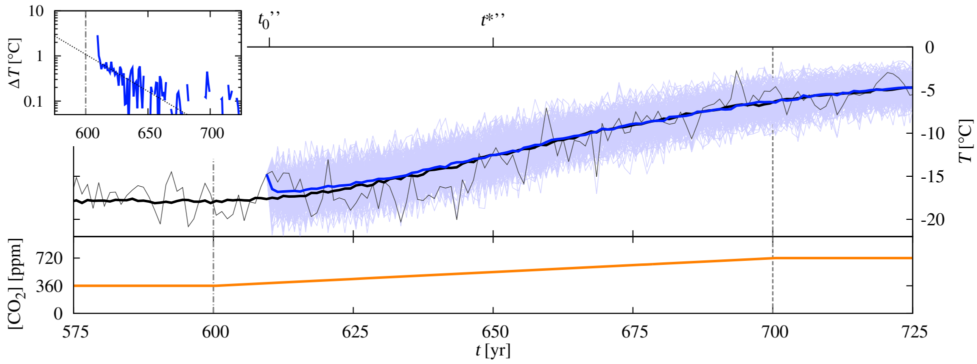

Fig. 2. The annual mean surface temperature T of a single gridpoint in the Southern Pacific ocean (at 180°E and about 64°S) as a function of time: the dark blue [dark gray in print] thick line marks the ensemble average taken over the ensemble initialized at $t_0'' = 610\,\text{yr}$, and the black thick line stands for the average on the attractor (represented numerically by an ensemble initialized at $t_0 = 0$); see the main text for details. The time evolutions of the 192 individual members of the newer ensemble (the third ensemble considered in this paper) are included in light blue [light gray in print], as thin lines. For better visibility, the last unperturbed year, which is year 609, is also included for the graphs of the newer ensemble. In this year, all members of the newer ensemble, and thus also their ensemble average, coincide with the originating member of the first ensemble. The time evolution of this member is indicated by a dark gray thin line. In the inset, we show the difference ΔT of the dark blue [dark gray in print] and the black thick lines as a function of time with an exponential fit for the interval $[613, 631]\,\text{yr}$ which yields a relaxation time $\tau \approx 27\,\text{yr}$. The CO_2 concentration, i.e., the forcing, is also included (in orange, in the bottom plot), as well as t_0'', and $t^{*\prime\prime}$ (the time instant up to which the convergence takes place). The vertical dot-dashed (dashed) line in gray marks the beginning (end) of the linear ramp in the CO_2 concentration.

which $t_c' \approx 5\tau$. This observation illustrates that the black line indeed corresponds to a snapshot attractor: it attracts any ensemble, whenever it is initialized, and it is unique. This is why the black line characterizes the climate for any t, even *along the CO_2 ramps*. In the period $t_0' < t < t^{*\prime}$ the average taken over the second ensemble (red line) differs from the average characterizing the climate (black line). We see here that ensemble results taken before convergence are, of course, misleading also in a period of climate change.

Figure 2 concerns a third ensemble which is initialized on the increasing ramp of the CO_2 concentration: we took one member of the first ensemble on the turn of year 609 to 610 (i.e., at $t_0'' = 610\,\text{yr}$)[5], and perturbed its surface pressure field by an amount on the order of 0.1 hPa. This way we obtained 192 approximate replicas of a weather situation (that of $t = 610\,\text{yr}$ in the chosen ensemble member) occurring practically *on* the snapshot attractor. Letting our third ensemble evolve in time is done in the very same spirit as what is performed by [10]; this way we simulate how the internal variability emerges from approximate observational data corresponding to a particular time instant.

In Figure 2, showing the annual mean surface temperature as a function of time in one particular arbitrarily chosen gridpoint of the Southern Pacific, one can observe how the different members of the third ensemble (in light blue [light gray in print]) spread out from one point, corresponding to year 609. By $t^{*\prime\prime} \approx 650\,\text{yr}$, the ensemble

[5] Note that our calendar starts with year 0.

average taken over the third ensemble (dark blue line [dark gray in print]) converges to that taken over the natural distribution of the snapshot attractor (i.e., to the black line)[6], thus $t''_c \approx 40\,\mathrm{yr}$ here.[7] A precise estimation of a relaxation time τ is, in this case, more difficult, but we estimate it to be $\tau \approx 27\,\mathrm{yr}$. It is clear that from $t^{*\prime\prime} \approx 650\,\mathrm{yr}$ on, i.e., after a "fast" (exponential-like) convergence, the third ensemble (which emerges from realistic initial conditions, i.e., from those that are on the snapshot attractor but are localized to some particular region of the attractor only) also represents the natural probability distribution.

As a main consequence which illustrates the importance of the convergence, the dark blue line [dark gray in print] of Figure 2 represents the response of the system to the increasing CO_2 concentration faithfully for $t > t^{*\prime\prime} \approx 650\,\mathrm{yr}$ (i.e., after a merger with the black line), but *not* earlier.

3 General conclusions

The results in Section 2 suggest that a similar investigation of the convergence to the natural probability distribution of the attractor would be informative also in any GCM. More generally, this strategy is to be followed in the investigation of any dissipative dynamical system with drifting parameters, both high- and low-dimensional ones, since all such systems call for a description in terms of snapshot attractors and ensembles.

High-dimensional systems, like GCMs, have, however, several different time scales for relaxation; it has been reported e.g. in [10] that: the time scales determined by the atmosphere, the land and the sea ice are short (up to years), those by the upper ocean are on the order of several decades, while those by the abyssal circulation are on the order of thousands of years. The deep ocean was, at the same time, observed by the authors to change very little during the investigation period of a few centuries. In this case, we believe, it might be useful to consider the abyssal circulation "frozen-in" rather than aiming to explore its own internal variability. If the abyssal circulation proves to be approximately the same in all members of the ensemble, it is meaningful to concentrate on the internal variability of the rest of the system.

Viewing of a slow dynamics to be "frozen-in" can be relaxed in the following spirit. The described situation with a large time scale separation appears and is exploited also in other research areas (e.g., that of subgrid-scale process parametrizations). Such systems are called "fast-slow systems" [26]. The standard approach separates variables (possibly after an appropriate coordinate transformation) into two groups: x representing fast and y representing slow variables. Just the opposite to parametrizations, these are the fast x variables that are of (primary) interest regarding climate change. Regarding y as a parameter a traditional attractor can be defined in the x-space; and in case of a time evolution of y independent of x (obtained in the spirit of a so-called deterministic parametrization) a snapshot attractor is present in the x-space. In our example the abyssal ocean dynamics can be represented by the variables y, and the rest of the system (including the upper ocean) by the variables x.

[6] We represent the snapshot attractor and its natural distribution in this case by an ensemble initialized at $t_0 = 0$ and consisting of 192 members – we have 152 additional members beyond those constituting the ensemble discussed in Figure 1, i.e., the first ensemble of the paper.

[7] It is illuminating to see the drastic difference between an ensemble property (the thick blue line [dark gray in print]) and the single time series (the dark gray thin line) of the chosen originating member. The latter oscillates about the average over the snapshot attractor (the thick black line) in the entire time span shown, while the former rapidly converges to this average. This illustrates the clear advantage of the ensemble view.

However, beside the y terms, in the equations for x genuine external forcing terms may also be present that have other time scales than y, such as e.g., industrial CO_2 emissions or a volcano eruption.

Having numerically constructed the said snapshot attractor for the x variables alone for the investigation period, the convergence of the ensemble initialized at some t_0 to the natural probability distribution can be studied in detail. We emphasize that for such an investigation it suffices to perturb any subset of the x variables, e.g., the atmospheric variables only (without the upper ocean), as already done in our examples and in [10]. This would allow identifying the time t^* from which on the plethora of the ensemble characterizes appropriately the probabilities of all possibilities that are permitted by the climate dynamics. As discussed in Section 1, the further evolution of the ensemble, i.e., for $t > t^* = t_0 + t_c$, reflects accurately any changes in the climate, i.e., in the snapshot attractor and its natural distribution, due to the used external forcing.

The investigation described here may also be necessary to carry out in any climate model, in order to obtain solid knowledge and correct results about the climate change in the particular model that is subject to some particular forcing scenario. Our method for numerically determining the forced response eliminates the uncertainty of any single-realization climate projection that originates from the internal variability of the dynamics. Instead, it offers an opportunity to study also the dynamical structure and the time evolution of the internal variability itself.

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