Vector-valued modular forms and generalized Moonshine

Dubrovnik, June 2009

Introduction

Modular forms: holomorphic functions $f\colon \mathbf{H} \to \mathbb{C}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$ and some (even integer) number w.

$f(\tau)$ is periodic $\Rightarrow$ q-expansion (with $q = e^{2\pi i\tau}$)

$$f = \sum_{n\in\mathbb{Z}} f_n q^n$$

Classically: holomorphicity at $\tau = i\infty$.

More generally: finite order pole.

Example: Eisenstein series (for $k > 1$)

$$E_{2k}(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k\text{-}1}(n)\, q^n$$

where

$$\sigma_k(n) = \sum_{d|n} d^k$$

and B_k is the k-th Bernoulli number.

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k}\, E_{2k}(\tau)$$

Any holomorphic modular form is a polynomial in E_4 and E_6.

Examples: $E_8 = E_4^2$ and $E_{10} = E_4 E_6$.

Discriminant form (weight 12)

$$\Delta = \frac{1}{1728}\left(E_4^3 - E_6^2\right) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Doesn't vanish on **H**!

Hauptmodul (weight 0)

$$J(q) = \frac{E_4^3}{\Delta} - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots$$

Univalent map $J : X(1) \to \mathbb{CP}^1$.

Any modular form of weight 0 is a polynomial in $J(q)$.

Applications:

- function theory
- algebraic geometry (STWW)
- number theory (CFT)
- combinatorics (partitions)
- topology (elliptic cohomology)

- algebra (VOA-s)

- physics (string theory and 2d CFT)

- Moonshine

Need for a theory of vector-valued modular forms.

Vector-valued modular forms

Holomorphic maps $\mathbb{X}:\mathbf{H}\to\mathbb{C}^d$ such that

$$\boxed{\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^w\,\rho\begin{pmatrix}a\,b\\c\,d\end{pmatrix}\mathbb{X}(\tau)}$$

ρ is a suitable (projective) representation of $\mathrm{SL}_2(\mathbb{Z})$ (more precisely, an ordinary representation of $\boldsymbol{B}_3$).

Exponent matrix: $\exp(2\pi i\mathbf{\Lambda})=\rho\begin{pmatrix}1\,1\\0\,1\end{pmatrix}$.

q-expansion:

$$\exp(-2\pi i\mathbf{\Lambda}\tau)\,\mathbb{X}(\tau)=\sum_{n\in\mathbb{Z}}\mathbb{X}[n]\,q^n$$

$\mathbb{X}$ meromorphic at $\tau=i\infty$ if singular part $\mathcal{P}\mathbb{X}=\sum_{n<0}\mathbb{X}[n]\,q^n$ is a finite sum.

Thanks to Δ, general case may be reduced to $w=0$, in which case ρ is a true representation of $\mathrm{PSL}_2(\mathbb{Z})$.

$\mathcal{M}(\rho)$: linear space of weight 0 forms.

Singular part map $\mathcal{P}: \mathcal{M}(\rho) \to \mathbb{C}[q^{-1}]^d$ is affected by choice of $\mathbf{\Lambda}$.

Is there a "best" choice?

Trace condition: $\mathcal{P}$ is bijective if and only if

$$\boxed{\mathrm{Tr}(\mathbf{\Lambda}) = d - \frac{\alpha}{2} - \frac{\beta_1 + 2\beta_2}{3}}$$

$d, \alpha, \beta_1, \beta_2$: nonnegative integers (signature of ρ).

The fundamental matrix

Multiplication by $J(q)$ takes $\mathcal{M}(\rho)$ to itself $\Rightarrow$

$$\boxed{\mathcal{M}(\rho) \text{ is a } \mathbb{C}[J]\text{-module (of finite rank)}}$$

Fundamental matrix $\Xi(q)$:

1. Columns generate $\mathcal{M}(\rho)$;

2. $\Xi(q) \to q^{\mathbf{\Lambda}\text{-}1}$ as $q \to 0$.

Exists and is unique provided $\mathcal{P}$ is bijective.

The inversion formula

Determinantal formula:

$$\det \Xi(q) = \left(\frac{E_4(q)}{\Delta(q)^{1/3}}\right)^{\beta_1+2\beta_2} \left(\frac{E_6(q)}{\Delta(q)^{1/2}}\right)^{\alpha}$$

$\Rightarrow$ $\Xi(q)$ is invertible (except at elliptic points).

Transformation rule:

$$\Xi\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Xi(\tau)$$

$\Rightarrow$ each component of $\mathbb{J}\mathbb{X} = \Xi(q)^{-1}\mathbb{X}(q)$ is a polynomial in $J(q)$.

Three alternative representations:

1. polynomial representation $\mathcal{J}\mathbb{X} = \boldsymbol{\Xi}(q)^{-1}\,\mathbb{X}(q)$;

2. q-expansion $\mathbb{X}(q) = \sum_{n\in\mathbb{Z}} \mathbb{X}[n]\,q^{\boldsymbol{\Lambda}+n}$;

3. singular part $\mathcal{P}\mathbb{X} = \sum_{n<0} \mathbb{X}[n]\,q^{n}$.

Inversion formula:

$$\mathcal{J}\mathbb{X}(w) = \frac{1}{2\pi i}\oint \frac{J'(q)}{w-J(q)}\boldsymbol{\Xi}(q)^{-1}\,q^{\boldsymbol{\Lambda}}\mathcal{P}\mathbb{X}(q)\,dq$$

The compatibility equation

Should we know $\Xi(q)$, we would know everything about $\mathcal{M}(\rho)$.

How can we determine $\Xi(q)$?

$\mathcal{M}(\rho)$ is a differential module: the (universal) differential operator

$$\nabla = \frac{E_{10}(q)}{\Delta(q)}\, q \, \frac{\mathrm{d}}{\mathrm{d}q}$$

maps $\mathcal{M}(\rho)$ to itself $\Rightarrow$ the matrix entries of

$$\mathcal{D}(q) = \Xi(q)^{-1}\, \nabla \Xi(q)$$

are (first order) polynomials in $J(q)$.

$$\boxed{\mathcal{D}(q) = J(q)\,(\mathbf{\Lambda} - 1) + \mathbf{\Lambda}\mathcal{X} - (\mathcal{X} + 240)(\mathbf{\Lambda} - 1)}$$

Characteristic matrix ("constant part" of Ξ):

$$\mathcal{X} = \lim_{q \to 0} \left(q^{-\mathbf{\Lambda}} \mathbf{\Xi}(q) - q^{-1} \right)$$

The pair $(\mathbf{\Lambda}, \mathcal{X})$ characterizes the representation ρ.

Compatibility equation

$$\boxed{\nabla \mathbf{\Xi}(q) = \mathbf{\Xi}(q)\, \mathcal{D}(q)}$$

First order (Fuchsian) ODE !

Boundary condition: existence of $\mathcal{X}$.

May be solved, e.g. recursively.

How can we determine $\mathcal{X}$?

1. Rademacher-Petersson formula

$$\mathcal{X}_{ij} = 2\pi\sqrt{\frac{1-\boldsymbol{\Lambda}_{jj}}{\boldsymbol{\Lambda}_{ii}}}\sum_{n=1}^{\infty}\frac{1}{n}I_1\left(\frac{4\pi}{n}\sqrt{\left(1-\boldsymbol{\Lambda}_{jj}\right)\boldsymbol{\Lambda}_{ii}}\right)\mathcal{S}_{ij}(n)$$

Problems: poor convergence, arithmetic properties, computation of Kloosterman-sums.

2. Invariant theory

Polynomial invariants (covariants) of $\rho \to$ polynomial relations between the matrix elements of $\Xi(q)$ and $\mathcal{X}$.

Problem: hard to compute.

3. Spectral condition

The matrices

$$\mathcal{A} = \frac{31}{36}(1-\boldsymbol{\Lambda}) - \frac{1}{864}(\mathcal{X}+\boldsymbol{\Lambda}\mathcal{X}-\mathcal{X}\boldsymbol{\Lambda})$$

and

$$\mathcal{B} = \frac{41}{24}(1-\boldsymbol{\Lambda}) + \frac{1}{576}(\mathcal{X}+\boldsymbol{\Lambda}\mathcal{X}-\mathcal{X}\boldsymbol{\Lambda})$$

satisfy the monodromy equation

$$\boxed{\mathcal{A}(\mathcal{A}-1)=\mathcal{B}(\mathcal{B}-1)(\mathcal{B}-2)=0}$$

System of quadratic equations for $\mathcal{X}$ (given $\boldsymbol{\Lambda}$).

Overdetermined for $d < 6$.

Example: the Ising model ($c = \frac{1}{2}$). Signature: $(3,1,1,1)$.

$$\mathbf{\Lambda} = \frac{1}{48}\begin{pmatrix} 47 & & \\ & 23 & \\ & & 2 \end{pmatrix} \qquad \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$$q^{-\mathbf{\Lambda}}\mathbf{\Xi}(q) = q^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2325 & 94208 \\ 1 & 275 & -4096 \\ 1 & -25 & -23 \end{pmatrix} +$$

$$+\, q\begin{pmatrix} 1 & 60630 & 9515008 \\ 1 & 13250 & -1130496 \\ 1 & -4121 & 253 \end{pmatrix} + q^2\begin{pmatrix} 1 & 811950 & 356765696 \\ 1 & 235500 & -63401984 \\ 1 & -102425 & -1794 \end{pmatrix} + \dots$$

Moonshine

McKay: $196884 = 196883 + 1$.

Conway and Norton: Monstrous Moonshine.

Frenkel, Lepowsky and Meurman: Moonshine module.

Borcherds: proof of the Moonshine conjectures.

Norton: generalized Moonshine.

Dixon, Ginsparg and Harvey: relation to orbifold CFT.

Generalized Thompson-McKay series: $Z(g,h|\tau)$ is the graded trace of h in the g-twisted module ($gh = hg$).

1. $Z(g^z, h^z|\tau) = Z(g,h|\tau)$ for all $z \in \mathbb{M}$;

2. $Z\left(g^{-1}, h^{-1}|\tau\right) = Z(g,h|\tau)$;

3. Equivariance

$$\boxed{Z\left(g,h\left|\frac{a\tau+b}{c\tau+d}\right.\right) = \alpha Z\left(g^a h^b, g^c h^d|\tau\right)}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and some cocycle factor α;

4. $Z(g,h|\tau)$ is either constant or a Hauptmodul.

Replication

n-fold symmetric product: orbifold with respect to all permutations of n identical copies of the Moonshine module.

Automorphism group contains the Monster (diagonal action) $\rightsquigarrow$ higher Thompson-McKay series $Z^{(n)}(g,h|\tau)$.

$$Z^{(n)}(g,h|\tau) = \frac{1}{n!} \sum_{\substack{xy=yx \\ x,y\in S_n}} \prod_{\xi\in\mathcal{O}(x,y)} Z\left(g^{\lambda_\xi}, g^{\kappa_\xi}h^{\mu_\xi} \middle| \frac{\mu_\xi\tau+\kappa_\xi}{\lambda_\xi}\right)$$

$\lambda_\xi, \mu_\xi, \kappa_\xi$: integers characterizing the orbit $\xi\in\mathcal{O}(x,y)$.

Replication: $Z^{(n)}(g,h|\tau)$ is a polynomial in $Z(g,h|\tau)$.

For each n, the $Z^{(n)}(g,h|\tau)$ form a vector-valued modular form, with same automorphy factor as $Z(g,h|\tau)$.

But: in this monomial basis the Dehn-twist is not diagonal $\rightsquigarrow$ change to primary basis.

$$Z^{(n)}(g,h|\tau) = \sum_i Z_i^{(n)}(\tau)\,\psi_i(g,h)$$

By general teory, there exists $\mathcal{Z}_i^{(n)}(J) \in \mathbb{C}[J]$ such that

$$\boxed{Z^{(n)}(g,h|\tau) = \sum_i \mathcal{Z}_i^{(n)}(J)\,\zeta_i(g,h|\tau)}$$

for some 'elementary' Thompson-McKay series $\zeta_i(g,h|\tau)$.

Differential equation

$$\nabla \zeta_i(g,h|\tau) = \sum_j \mathcal{D}_{ji}(\tau)\, \zeta_j(g,h|\tau)$$

Boundary condition: behavior of $\zeta_i(g,h|\tau)$ as $\tau \to i\infty$.

Product of two TM-series is again a TM-series (if no cocycle factors) $\Rightarrow$

$$\zeta_i(g,h|\tau)\, \zeta_j(g,h|\tau) = \sum_k C_{ij}^k(\tau)\, \zeta_k(g,h|\tau)$$

where the coefficients C_{ij}^k are polynomials in $J(\tau)$.

Example: $G = \mathbb{Z}_2$.

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & & & \\ & \tfrac{1}{2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$q^{-\mathbf{\Lambda}}\mathbf{\Xi}(q) = q^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -24 & 24 & 1 & 1 \\ 4096 & 276 & 0 & 0 \\ 98304 & -2048 & 0 & 0 \\ 98304 & -2048 & 0 & 0 \end{pmatrix} +$$

$$q\begin{pmatrix} 98580 & 2048 & 0 & 0 \\ 1228800 & 11202 & 0 & 0 \\ 10747904 & -49152 & 0 & 0 \\ 10747904 & -49152 & 0 & 0 \end{pmatrix} + q^2\begin{pmatrix} 10745856 & 49152 & 0 & 0 \\ 74244096 & 184024 & 0 & 0 \\ 432144384 & -614400 & 0 & 0 \\ 432144384 & -614400 & 0 & 0 \end{pmatrix} + \cdots$$

	ζ_1	ζ_2	ζ_3	ζ_4
$(1,1)$	$J(\tau)-24$	24	2	1
$(1,x)$	$T_{2B}(\tau)-24$	$T_{2A}(\tau)-T_{2B}(\tau)+24$	0	1
$(x,1)$	$T_{2A}\left(\frac{\tau}{2}\right)-T_{2B}\left(\frac{\tau}{2}\right)$	$T_{2B}\left(\frac{\tau}{2}\right)$	0	1
(x,x)	$T_{2A}\left(\frac{\tau+1}{2}\right)-T_{2B}\left(\frac{\tau+1}{2}\right)$	$T_{2B}\left(\frac{\tau+1}{2}\right)$	0	1

$T_{2A}=q^{-1}+4372q+96256q^2+\ldots$ and $T_{2B}=q^{-1}+276q-2048q^2+\ldots$

$\star$	ζ_1	ζ_2	ζ_3	ζ_4
ζ_1	$(J-24)\zeta_1-4096\zeta_2+98304\,(\zeta_3-\zeta_4)$			
ζ_2	$24\zeta_1-2048\zeta_3+4096\zeta_4$	$-\zeta_1+(J+552)\zeta_4$		
ζ_3	$(J-24)\zeta_3$	$24\zeta_3$	$2\zeta_3$	
ζ_4	ζ_1	ζ_2	ζ_3	ζ_4

Outlook

- vector-valued modular forms pretty much under control
- explicit computational procedures
- applications to Moonshine

and open questions

- automorphic forms for other groups
- Hecke operators
- Jacobi and Siegel forms