Introduction

Group theory is currently one of the most important mathematical disciplines, with manifold applications in

- mathematics (Galois theory, differential equations, geometry, etc.)
- physics (crystallography, solid state physics, high energy physics, gauge theories, phase transitions, general relativity, etc.)
- chemistry (molecular symmetries)

Question: what makes groups so ubiquitous?

Twofold origin of the group concept: notions of symmetry and (co)homology.

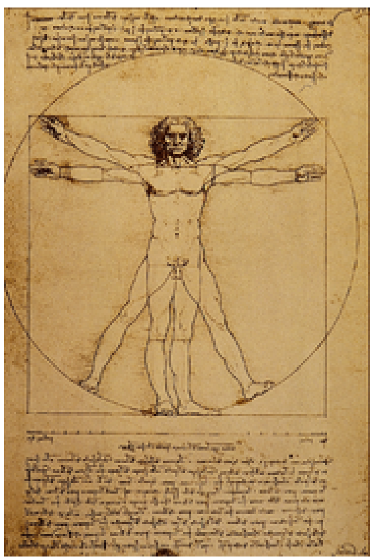
Homology is a topological notion characterizing the ”connectedness” of a manifold through a sequence of groups associated to it.

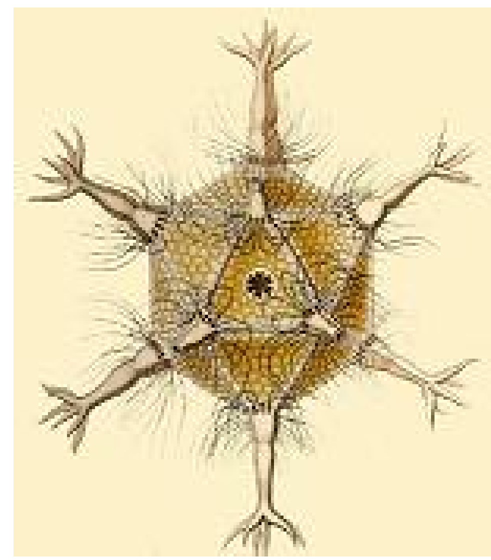
In physics, the notion of homology has important applications in the study of quantum systems, e.g. the Berry phase and topological quantum computing, gauge theories (instantons), general relativity (gravitational singularities) and string theory (branes).

The notion of homology can be extended to much more general situations (homological algebra).

Symmetry: invariance under suitable transformations.

A classical example is the bilateral (mirror) symmetry of the human body, characteristic of a very large class of animals, the Bilateralia (to be contrasted with the five-fold rotational symmetry of echinoderms and the icosahedral symmetry of certain micro-organisms).





Similar symmetry patterns show up in artificial (man made) objects like buildings, furniture, decorations, etc.

Symmetry transformation can be composed, and groups describe the algebra of symmetries, hence group theory provides techniques to convert qualitative information into quantitative one.

Example: counting of phenomenological constants.

Small deformations of an elastic medium are described by Hooke's law

$$\sigma_{ij} = L_{ijkl} u_{kl}$$

with u_{ij} and σ_{ij} denoting the deformation and the stress tensor.

$3^4 = 81$ different L_{ijkl} components, related by Onsager's reciprocity law

$$L_{ijkl} = L_{jikl} = L_{klij}$$

Question: how many independent coefficients L_{ijkl} (elastic moduli) characterize a given medium?

Answer: 2 for isotropic media, 21 for a triclinic crystal.

Explanation: crystalline structure characterized by its symmetries, part of which form a group of matrices, the so-called point group.

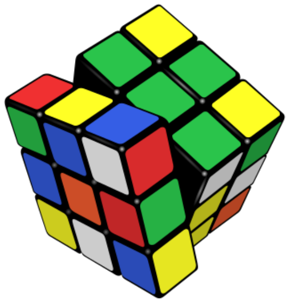
For a crystal with point group G, there are

$$\frac{1}{8}\sum_{g\in G}\left\{\mathrm{Tr}(g)^4+2\mathrm{Tr}(g)^2\,\mathrm{Tr}\big(g^2\big)+3\mathrm{Tr}\big(g^2\big)^2+2\mathrm{Tr}\big(g^4\big)\right\}$$

independent elastic moduli (in any dimension).

In complete generality, the scalar free energy is an invariant polynomial of the (symmetric) deformation tensor, hence the answer equals the number of independent invariants of the symmetrized square of the point group (the above formula gives the number of quadratic invariants, appropriate in the linear case described by Hooke's law).

Recreational mathematics: solving Rubik's Cube.



$43,252,003,274,489,856,000$ valid configurations.

Reach a given ('solved') position within shortest time ('speedcubing') or with the least number of moves.

Best results to date: 3.13 seconds (2023) and 16 moves (2019).

M. Davidson, J. Dethridge, H. Kociemba and T. Rokicki (2010) : all initial configurations can be solved in 20 moves or less, and some (in fact millions) actually require 20.

Elementary moves (6 in total): clockwise 90 degree rotation of a face.

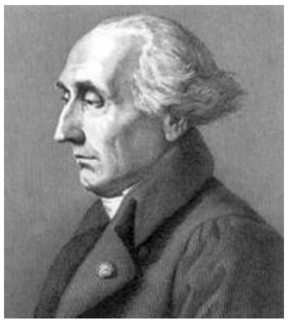
Each legal cube move can be obtained through a sequence of elementary moves, and solving the cube amounts to finding such a sequence.

Legal cube moves form a group (product being application of one after the other), the Cube group, whose elements are in one-to-one correspondence with the valid configurations: solving the cube amounts to finding a sequence of elementary moves whose product corresponds to the initial configuration.

Group theory helps develop efficient solver methods, and allows to associate Rubik's Cube with a quantum system of few particles.

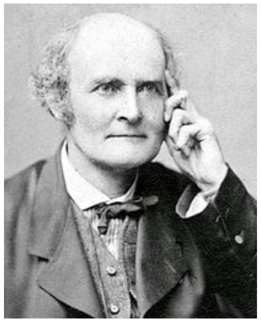
1 Historical highlights

Antiquity: application of symmetry principles in geometry (Euclid of Alexandria, Archimedes of Syracuse, etc.), classification of platonic solids.



J.-L. Lagrange (1771): solubility of polynomial equations.

É. Galois (1832): Galois theory.



A. Cayley (1854): abstract group concept.

É. Mathieu (1861,1873): discovery of the Mathieu groups.

F. Klein (1873): Erlangen program (classification of geometries via symmetry principles).

S. Lie (1871-1893): continuous transformation groups.

H. Poincaré (1882): homology groups, uniformization.

É. Picard (1883): differential Galois theory.

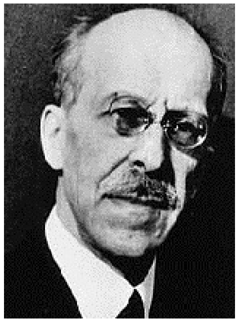
D. Hilbert (1888): theory if invariants, homological algebra.

  W. Killing (1880-1890) and É. Cartan (1894):

classification of simple Lie-groups and their Lie-algebras.

 G.F. Frobenius (1896): representations, group characters.

 W. Burnside (1903): finite groups.



I. Schur (1904): projective representations.

A. Haar (1933): invariant integrals.

1966-1976: sporadic simple groups.

Group theory in physics

Renaissance: symmetry principles in statics ("Epitaph of Stevinus").

L. Euler (1765): movement of rigid bodies.

E.S. Fedorov (1891), L. Schönflies (1891) and W. Barlow (1894): classification of crystal structures in 3D.

H. Poincaré (1900): symmetries of Maxwell's equations.

E. Noether (1915): symmetries vs conservation laws.

E. Wigner: symmetries in quantum physics (1933), classification of relativistic wave equations (1947).

C.N. Yang és R. Mills (1954): local gauge symmetries.

M. Gell-Mann (1963): "eightfold way" (basis of the quark model).

D. Shechtman (1982): discovery of quasi-crystals.

2 Fundamental concepts

Question: What is a group? How to compare groups?

A group is a set of elements with a suitable binary operation.

Binary operation: rule that assigns to two elements of a set a well-defined third element of that same set.

Examples: addition and multiplication of (integer, rational, real, complex, hypercomplex, p-adic, etc.) numbers, greatest common divisor and lowest common multiple of integers, addition and cross product of vectors (but not their scalar product), addition and multiplication of linear operators /polynomials/matrices.

Multiplicative infix notation: for a binary operation on the set X, we denote by $x \star y$ (or simply by xy) the element obtained by applying the operation to the elements $x, y \in X$.

A binary operation (on the set X) is

associative, if for any $x, y, z \in X$

$$x(yz) = (xy)z$$

commutative, if for any $x, y \in X$

$$xy = yx$$

unital, if there exists $\mathbf{1}_X \in X$ (the identity) such that for all $x \in X$

$$\mathbf{1}_X x = x \mathbf{1}_X = x$$

A **group** is a set G of elements together with an associative and unital binary operation (the 'product'), such that for all $x \in G$ there exists $x^{-1} \in G$ (the inverse of x) for which $xx^{-1} = x^{-1}x = \mathbf{1}_G$.

The order of a group is the cardinality of its set of elements. A group is a finite if its order is finite, i.e. a positive integer.

A group is said to be abelian if its product is commutative.

Abelian groups have very special properties (they form a subvariety of the algebraic variety of groups).

The groups G and H are isomorphic, denoted $G \cong H$, if there exists a bijective (i.e. one-to-one) map $\phi : G \to H$ that preserves products, i.e.

$$\phi(xy) = \phi(x)\,\phi(y)$$

for all $x, y \in G$ (such a map is called an isomorphism).

The relation of being isomorphic is reflexive ($G \cong G$), symmetric ($G \cong H$ implies $H \cong G$) and transitive ($G \cong H$ and $H \cong K$ implies $G \cong K$).

Isomorphy principle: isomorphic groups cannot be distinguished from each other by algebraic means (they have the same algebraic structure).

Group theoretic properties are the same for isomorphic groups, e.g. the orders of isomorphic groups are the same: $G \cong H$ implies $|G| = |H|$.

An automorphism of a group is an isomorphism of the group with itself, and the collection $\mathrm{Aut}(G)$ of all automorphism of a group G is itself a group, the automorphism group ('symmetry group') of G, with product the composition of maps.

A collection H of elements of a group G is called a subgroup, denoted $H < G$, if it contains the identity element of G and the product of any two of its elements, as well as the inverse of all its elements.

The relation of being a subgroup is an ordering. In particular, every subgroup is a group, and a subgroup of a subgroup is itself a subgroup: if $K < H$ and $H < G$, then $K < G$.

Generalizations of the group concept:

- relaxing the existence of inverses leads to monoids (with applications to automata theory & linguistics, renormalization, etc.);
- relaxing the associativity of the product results in quasi-groups (combinatorial applications like latin squares, aka. sudoku);
- a partially defined product gives groupoids (with applications to topology, the description of quasi-crystals, etc.);
- theoretical physics $\rightsquigarrow$ quantum groups, supergroups, ...

3 Examples of groups

3.1 Additive groups of numbers

Consider the set $\mathbb{Z}$ of all integer numbers with the binary operation of addition, which is

associative: $a+(b+c)=(a+b)+c$;

commutative: $a+b=b+a$;

unital: $a+0=0+a=a$

for all $a, b, c \in \mathbb{Z}$.

Every integer has an additive inverse, i.e. to each $a \in \mathbb{Z}$ corresponds its negative $-a$ such that $a+(-a)=(-a)+a=0$, hence the integers form an abelian (i.e. commutative) group $(\mathbb{Z},+)$, the additive group of integers.

Similar considerations apply to other sets of numbers, like the rationals $\mathbb{Q}$, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, etc., leading to the additive groups $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{C},+)$, and so on.

Remark: for an arbitrary integer n, the set $n\mathbb{Z}=\{nx \mid x \in \mathbb{Z}\}$ of integer multiples of n also forms a group $(n\mathbb{Z},+)$ with the operation of addition, which may be shown to be isomorphic with $(\mathbb{Z},+)$.

While the structure of $(\mathbb{Z},+)$ is relatively simple (it is an infinite cyclic group), the structure of the other additive groups is pretty complicated.

3.2 Integers modulo n (residue classes)

Euclid's division lemma. *Given integers a and $b \neq 0$, there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < |b|$, where $|b|$ denotes the absolute value of b.*

In the above setting, r is called the remainder of a upon division by b (while q is the quotient), and it is usually denoted as $r = a \bmod b$ (modulo operation).

$$(a_1 + a_2) \bmod b = ((a_1 \bmod b) + (a_2 \bmod b)) \bmod b$$

consequently the remainder upon division by b of $a_1 + a_2$ only depends on the remainders of a_1 and a_2.

For a positive integer $n > 1$ (called the modulus), let's consider the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with the binary operation that assigns to $x, y \in \mathbb{Z}_n$ the remainder upon division by n of their sum $x+y$, i.e. $(x+y) \bmod n$ (addition mod n).

This is an associative, commutative and unital operation, such that every $x \in \mathbb{Z}_n \setminus \{0\}$ has an inverse equal to $n-x$ (while the inverse of 0 is itself), hence $\mathbb{Z}_n$ becomes a finite abelian group of order n with this operation, the additive group of integers modulo n.

To each element of $k \in \mathbb{Z}_n$ corresponds the set $n\mathbb{Z}+k = \{nx+k \mid x \in \mathbb{Z}\}$, the collection of all integers that have remainder k upon division by n (a residue class mod n).

3.3 Matrix groups

A matrix A is a rectangular array of numbers, with the matrix element A_{ij} denoting the number found at the intersection of the $i^{\mathtt{th}}$ row and $j^{\mathtt{th}}$ column (in general, the elements of a matrix may come from any ring). A square matrix of size n has n rows and columns.

One can add matrices A and B of the same shape (i.e. with the same number of rows and columns) element-wise, that is $(A+B)_{ij}=A_{ij}+B_{ij}$, while their product AB is only defined if the number m of columns of A equals the number of rows of B, in which case it has matrix elements $(AB)_{ij}=\sum\limits_{k=1}^{m} A_{ik}B_{kj}$.

For a positive integer n, the identity matrix $\mathbf{1}_n$ of size n has diagonal matrix elements equal to 1, and all other equal to 0, i.e. $(\mathbf{1}_n)_{ij} = \delta_{ij}$.

Since the determinant of the identity matrix is 1, and the determinant of a product is the product of the determinants, $\det(AB) = \det(A)\det(B)$, a square matrix A is invertible, i.e. there exists a square matrix B such that $AB = BA = \mathbf{1}$, iff its determinant has a multiplicative inverse.

For any number ring R (collection of numbers containing 1 and closed under addition, subtraction and multiplication, e.g. the integers $\mathbb{Z}$ or the complex numbers $\mathbb{C}$) and a positive integer n, the collection of all invertible n-by-n matrices with entries from R with the operation of matrix product form a group $\mathsf{GL_n}(R)$, the general linear group over R.

$\mathsf{GL_n}(R)$ is infinite if R is, and is not abelian unless $n=1$.

Besides the general linear group $\mathsf{GL_n}(R)$, several of its subgroups play an important role in applications:

1. those matrices in which each row and column contains exactly one nonzero entry (monomial matrices) form a group $\mathsf{M}_n(R) < \mathsf{GL_n}(R)$;
2. the set $\Pi_n = \{A \in \mathsf{M}_n(R) \mid A_{ij} \neq 0 \text{ implies } A_{ij} = 1\}$ of permutation matrices is a subgroup of $\mathsf{M}_n(R)$, hence of $\mathsf{GL_n}(R)$;
3. the set

$$\Delta_n(R) = \{A \in \mathsf{GL_n}(R) \mid A_{ij} = 0 \text{ if } i \neq j\}$$

of diagonal matrices is an abelian subgroup of $\mathsf{M}_n(R)$;

Remark. Note that every monomial matrix is the product of a diagonal and a permutation matrix.

4. denoting by A^{T} the *transpose* of the matrix A, i.e. the matrix with entries $(A^{\mathsf{T}})_{ij} = A_{ji}$, the set

$$\mathsf{O}_n(R) = \left\{ A \in \mathsf{GL_n}(R) \mid A^{-1} = A^{\mathsf{T}} \right\}$$

of orthogonal matrices is a group. More generally, denoting by $\eta_{p,q}$ the diagonal matrix having the first p diagonal entries equal to 1, and the remaining q entries equal to -1 (the metric tensor of flat space with p spacelike and q timelike directions),

$$\mathsf{O}_{p,q}(R) = \left\{ A \in \mathsf{GL_{p+q}}(R) \mid A^{-1} = \eta_{p,q}^{-1} A^{\mathsf{T}} \eta_{p,q} \right\}$$

is also a group;

5. the set

$$\mathsf{Sp}_{2n}(R) = \left\{ A \in \mathsf{GL}_{2n}(R) \mid A^{-1} = \mathtt{J}_n^{-1} A^{\mathsf{T}} \mathtt{J}_n \right\}$$

of symplectic matrices is a group, where $\mathtt{J}_n$ is the block-diagonal matrix made up of n copies of the *Pauli-matrix*

$$\mathtt{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

6. those subsets of all the above whose elements satisfy $\det A = 1$, e.g. the special linear and orthogonal groups

$$\mathsf{SL}_n(R) = \{A \in \mathsf{GL}_n(R) \mid \det A = 1\}$$
$$\mathsf{SO}_n(R) = \{A \in \mathsf{O}_n(R) \mid \det A = 1\}$$

3.4 Symmetric and alternating groups

A permutation is a bijective self-map of a set onto itself ('reshuffling').

The product of permutations is the composition of the corresponding maps: this is an associative and unital binary operation, with the trivial permutation $\mathbf{id}_X\colon x \mapsto x$ as identity element.

Inverse of a permutation: inverse map (once again bijective).

The collection of all permutations of a (finite) set X forms a group $\mathsf{Sym}(X)$, the symmetric (*NOT symmetry*!) group over X.

$\mathsf{Sym}(X)$ is not commutative in case $|X|>2$, and has order

$$\left|\mathsf{Sym}(X)\right| = |X|!$$

Transposition: interchange of two elements.

Any permutation can be decomposed (in many ways) into a product of transpositions: while their number may change, whether there is an even or odd number of them characterizes the permutation, allowing to distinguish odd and even permutations.

The product of even permutations is even, hence even permutations form themselves a group $\mathsf{Alt}(X)$, the alternating group over X.

The symmetric groups $\mathsf{Sym}(X)$ and $\mathsf{Sym}(Y)$ are isomorphic precisely when $|X| = |Y|$, hence it is enough to consider the symmetric groups $\mathbb{S}_n = \mathsf{Sym}(\{1, \ldots, n\})$ (resp. alternating groups $\mathbb{A}_n = \mathsf{Alt}(\{1, \ldots, n\})$) of degree n, isomorphic to Π_n and $\mathsf{S}\Pi_n$ respectively.

A cycle (orbit) of a permutation is a set of points that are taken into each other by successive applications of the permutation (a fixed point is a cycle of length 1).

The cycles of a permutation $\pi \in \mathsf{Sym}(X)$ partition the set X, i.e. any two are either equal or disjoint, and each point of X belongs to some cycle.

A permutation is called cyclic if it has only one cycle of length greater than one, and the length of this cycle is its order (or period).

Every permutation can be decomposed into a product of cyclic ones.

A cyclic permutation $\pi \in \mathsf{Sym}(X)$ is uniquely determined by the sequence

$$x_1, x_2 = \pi(x_1), \ldots, x_n = \pi(x_{n-1}), \ldots$$

where x_1 is any element of the non-trivial cycle of π (cycle notation).

3.5 Geometric symmetry groups

Rigid motion: mapping of Euclidean space onto itself that preserves the distance of points (Euclidean isometry).

Types of rigid motions: translations, rotations, reflections, and different composites of the above.

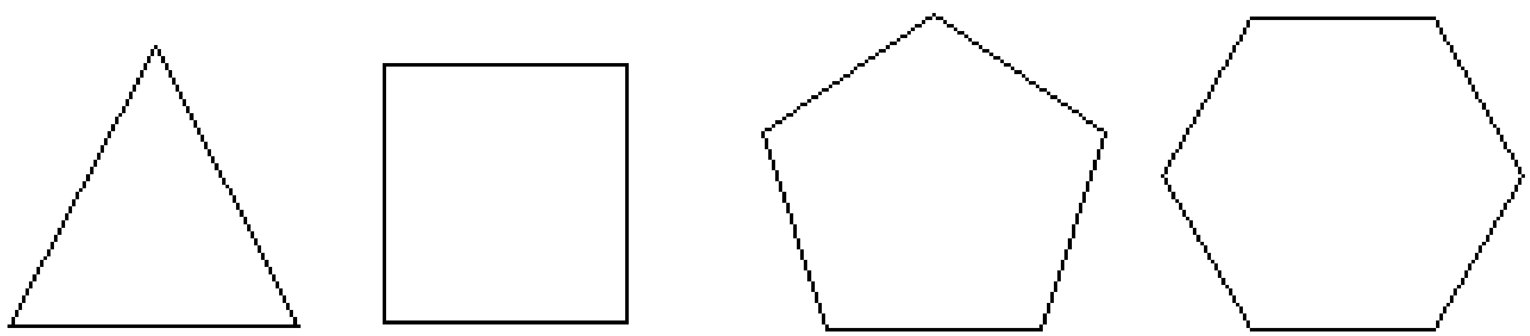
Symmetry of a geometric figure: rigid motion mapping the figure (as a set of points) onto itself.

For example, any rotation around an axis passing through the center of a sphere is a symmetry of the sphere (but the sphere has reflection symmetries as well).

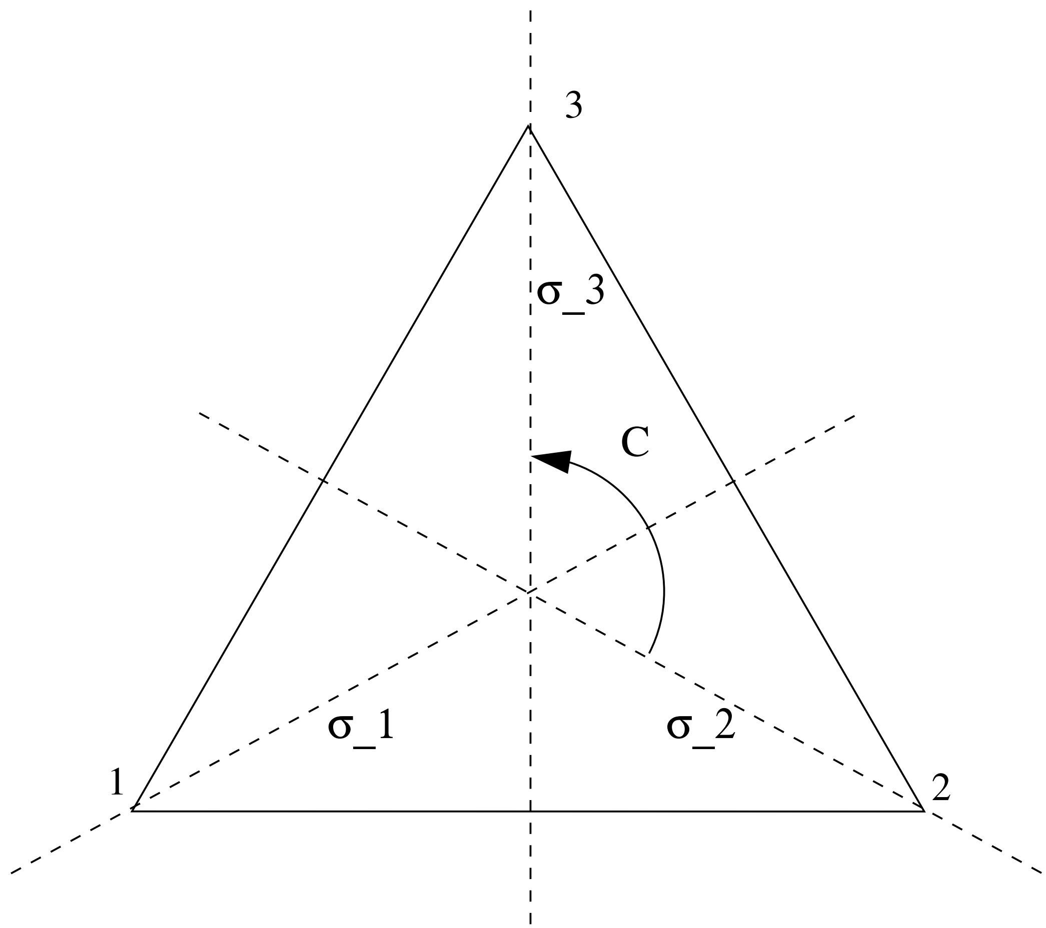
Regular polygon: convex plane figure all of whose sides are congruent (i.e. have equal length), and angles between neighboring sides are equal.

Medians (edge bisectors) of a regular polygon all meet in a single point, the center of the polygon.

For each integer $n > 2$ there is exactly one regular n-gon (up to similarity).



The symmetries of a regular n-gon, composed of rotations around the center (by multiples of $^{2\pi}/_{n}$) and reflections across lines passing through the center and some vertex, form the dihedral group $\mathbb{D}_n$ of degree n.



Since there are n different reflection axes (the medians for odd n, and the medians together with the diagonals passing through opposite vertices for even n) and n different rotations, the order of the dihedral group is $\left|\mathbb{D}_n\right| = 2n$.

For finite groups, the group structure can be neatly described using the Cayley table.

G	$\mathbf{1}_G$	$\dots$	h	$\dots$
$\mathbf{1}_G$	$\mathbf{1}_G$	$\dots$	h	$\dots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
g	g	$\dots$	gh	$\dots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

Since symmetries map the polygon onto itself, any set of distinguished subfigures (like vertices, edges, medians, etc.) is also mapped onto itself by a symmetry transformation.

As a consequence, each symmetry transformation induces a permutation of any chosen set of distinguished subfigures (be it vertices, edges, medians, etc.), which can prove handy for the computation of the Cayley table.

Remark: since the set V of vertices of a regular n-gon has cardinality n, the above correspondence is bijective only for $n=3$, when

$$\left|\mathsf{Sym}(V)\right| = n! = 2n = \left|\mathbb{D}_n\right|$$

$\mathbb{D}_3$	1	C	C^2	σ_1	σ_2	σ_3
1	1	C	C^2	σ_1	σ_2	σ_3
C	C	C^2	1	σ_3	σ_1	σ_2
C^2	C^2	1	C	σ_2	σ_3	σ_1
σ_1	σ_1	σ_2	σ_3	1	C	C^2
σ_2	σ_2	σ_3	σ_1	C^2	1	C
σ_3	σ_3	σ_1	σ_2	C	C^2	1

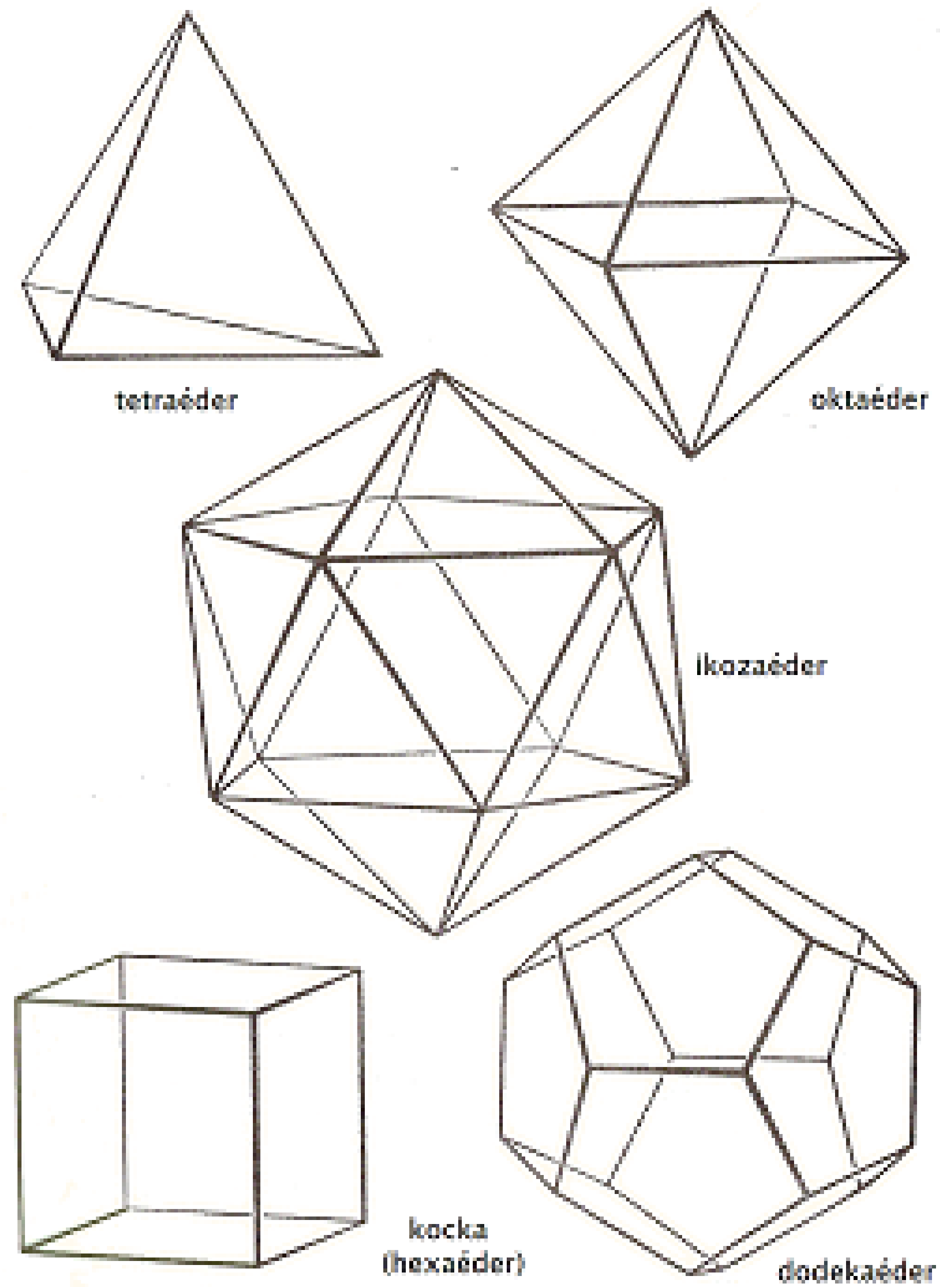
$()$
$(1,2,3)$
$(1,3,2)$
$(2,3)$
$(1,3)$
$(1,2)$

Multiplication table of $\mathbb{D}_3$ and the induced permutation of vertices.

A convex spatial figure all of whose bounding facets are congruent regular polygons is called a Platonic solid (regular polyhedron).

There are five different Platonic solids (up to similarity):

1. tetrahedron (4 triangles);
2. octahedron (8 triangles);
3. icosahedron (20 triangles);
4. cube (6 squares);
5. dodecahedron (12 pentagons).


tetraéder
oktaéder
ikozaéder
kocka
(hexaéder)
dodekaéder

The symmetry groups of regular polyhedra are:

- tetrahedral group $\mathbf{T} \cong \mathbb{A}_4$ (tetrahedron),
- octahedral group $\mathbf{O} \cong \mathbb{S}_4$ (octahedron $\rightleftharpoons$ cube)
- icosahedral group $\mathbf{I} \cong \mathbb{A}_5$ (icosahedron $\rightleftharpoons$ dodecahedron).

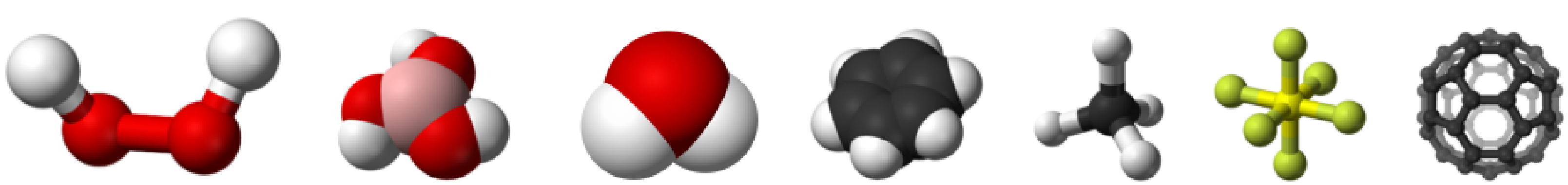
Regular convex polytopes in 4D: simplex (5-cell), orthoplex (16-cell), hypercube (8-cell), 600-cell, 120-cell, 24-cell (with 1152 symmetries).

Only 3 regular polytopes in dimensions >4: the simplex , the orthoplex (cross-polytope) and the hypercube.

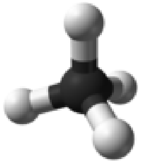
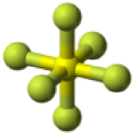
3.6 Molecular symmetry groups

Charge density in molecules invariant under a finite group (point group) of geometric transformations, leading to restrictions on

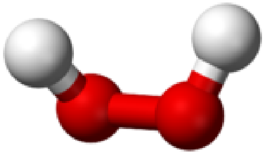
1. the structure of the molecular spectrum (Wigner, Tisza);
2. electromagnetic characteristics (dipole and magnetic moments);
3. chemical properties.

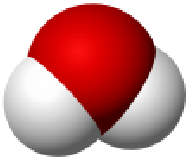


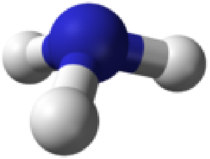
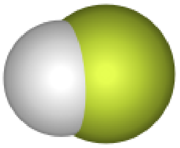
Point groups in 3D: 7 polyhedral groups

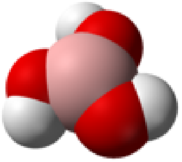
1. $\mathbf{T} \cong \mathbb{A}_4$ (chiral tetrahedral)
2. $\mathbf{T}_{\mathrm{d}} \cong \mathbb{S}_4$ (tetrahedral), e.g. methane 
3. $\mathbf{T}_{\mathrm{h}} \cong \mathbb{A}_4 \times \mathbb{Z}_2$ (pyritohedral)
4. $\mathbf{O} \cong \mathbb{S}_4$ (chiral octahedral)
5. $\mathbf{O}_{\mathrm{h}} \cong \mathbb{S}_4 \times \mathbb{Z}_2$ (octahedral), e.g. sulfur hexafluoride SF_6 
6. $\mathbf{I} \cong \mathbb{A}_5$ (chiral icosahedral)
7. $\mathbf{I}_{\mathrm{h}} \cong \mathbb{A}_5 \times \mathbb{Z}_2$ (icosahedral), e.g. C_{60} fullerene 

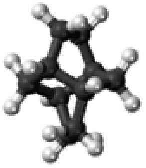
+ 7 infinite families of axial groups

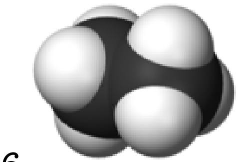
1. $C_n \cong \mathbb{Z}_n$ (cyclic), e.g. hydrogen peroxide  $(n=2)$

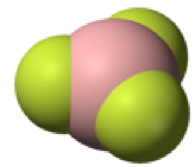
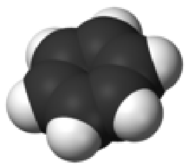
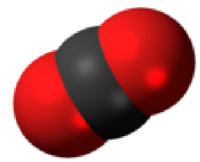
2. $C_{n\mathrm{v}} \cong \mathbb{D}_n$ (pyramidal), e.g. water  $(n=2)$,

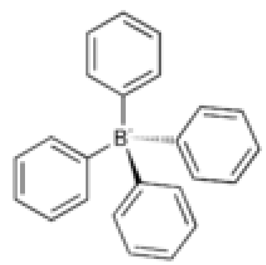
   ammonia  $(n=3)$, hydrogen fluoride  $(n=\infty)$

3. $C_{n\mathrm{h}} \cong \mathbb{Z}_n \times \mathbb{Z}_2$, e.g. boric acid  $(n=3)$

4. $D_n \cong \mathbb{D}_n$ (dihedral), e.g. twistane $\mathtt{C}_{10}\mathtt{H}_{16}$  $(n=2)$

5. $D_{n\mathrm{d}} \cong \mathbb{D}_{2n}$ (anti-prismatic), e.g. ethane $\mathtt{C_2H_6}$  $(n=3)$

6. $D_{n\mathrm{h}} \cong \mathbb{D}_n \times \mathbb{Z}_2$ (prismatic), e.g. boron trifluoride  $(n=3)$, benzene  $(n=6)$, carbon dioxide  $(n=\infty)$

7. $S_{2n} \cong \mathbb{Z}_{2n}$, e.g. tetraphenylborate  $(n=4)$

3.7 Crystalline symmetry groups

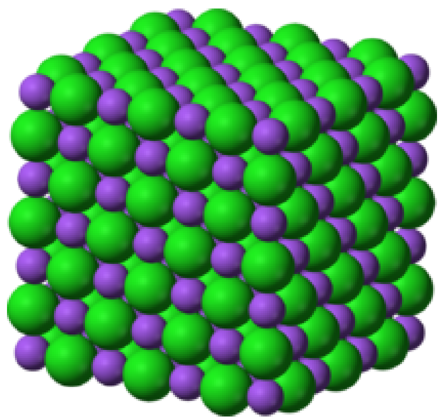
Some homogeneous substances exhibit anisotropic (direction dependent) behavior on a macroscopic scale: e.g. mechanical, optical, electric, etc. properties of crystals.

At the phenomenological level, the material characteristics (like permittivity, heat conductivity, elastic moduli, etc.) are tensorial quantities (rather than scalars).

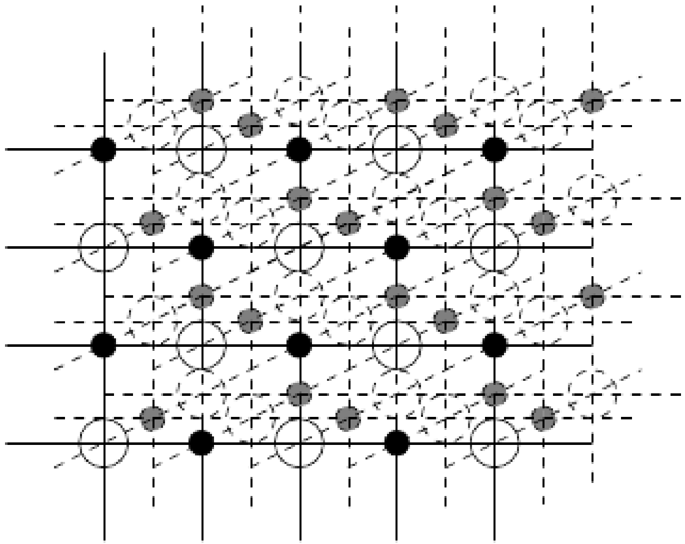
Since both microscopic homogeneity/isotropy and unordered microscopic inhomogeneity/anisotropy leads to isotropic behavior, macroscopic anisotropy is a consequence of ordered microscopic inhomogeneity/anisotropy.

Ordered microscopic anisotropy is realized in ferro- and ferrimagnetic materials, while ordered inhomogeinity in (quasi-)crystals.

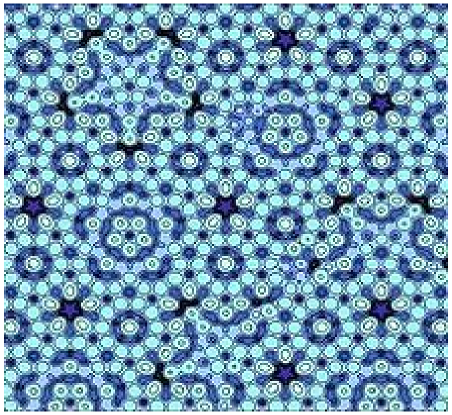
Macroscopic order can arise from discrete translational symmetries: a periodic structure in space exhibits ordered inhomogeinity.



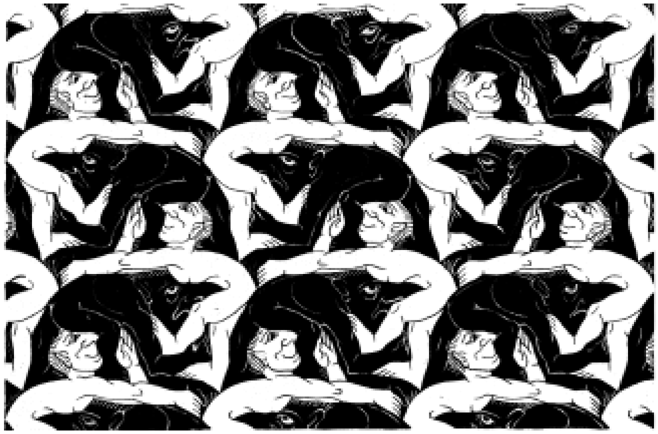
Crystalline substance: microscopic components (atoms/molecules/ions) distributed periodically in space, localized (in the absence of defects) around the lattice points of a 3D periodic lattice (the crystal lattice).



Quasi-crystal: aperiodic structure exhibiting long range order.



Space group: full symmetry group of the crystal, taking into account the symmetries of its microscopic constituents (composed of translations, rotations, reflections, inversions and various combinations of the above).



Translation subgroup: group of translations taking the crystal into itself.

Point group: finite group describing the rotational and reflection symmetries of the crystal.

Crystal structures are grouped into crystal classes, families and systems according to their point groups and translation subgroups.

Order of a transformation: smallest positive integer N such that the $N^{\mathtt{th}}$ power of the transformation is the identity.

Crystallographic restriction: the number of integers coprime to the order of any element of the point group cannot exceed the dimension of space (valid for periodic structures, not quasi-crystals).

dimension	$2,3$	$4,5$	$6,7$
allowed values	$\{2,3,4,6\}$	$\cup\{5,8,10,12\}$	$\cup\{7,9,14,18\}$

Consequence: there are only finitely many different crystal structures in any space dimension.

Classification results

dimension	2	$3^\dagger$	$4^\ddagger$	5^*	6^*
# crystal systems	4	7	33	59	251
# point groups	10	32	227	955	7104
# space groups	17	230	4894	222097	28934974

$\dagger$: Fedorov (1891), Schönflies (1891) and Barlow (1894).

$\ddagger$: Brown, Bülow and Neubüser (1978).

$*$: Plesken and Schulz (2000).

3.8 Space-time symmetries

Kinematics: description of the movement of material bodies, i.e. of the time evolution of their mutual emplacements (relative distances).

Frame of reference: system of bodies with known relative motions.

The relative motions of all bodies of the Universe are completely determined by their motions with respect to a specific frame of reference.

Question: are there reference frames that are more useful than others?

Answer: inertial frames, in which the inertial motion of isolated (not interacting with the rest of the Universe) bodies is uniform translation.

Remark: inertial frames move at constant speed relative to each other.

In classical (Newtonian) mechanics

- physical space is 3D Euclidean
- time is a 1D continuum
- forces act instantly, without any delay (action at a distance).

Galileo's relativity principle: not only the law of inertial motion, but all mechanical laws look the same in every inertial frame.

It follows that in inertial frames,

- both space and time are homogeneous, without a preferred origin;
- space is isotropic, i.e. there are no preferred directions;
- any reference frame obtained via a boost (constant speed uniform translation) from an inertial one is itself inertial.

These are universal symmetries governing the structure of natural laws: the laws of classical mechanics are the same anywhere and at anytime, irrespective of the orientation and of the inertial frame chosen.

The symmetries of classical mechanics form the Galilei group $\mathcal{G}$, whose elements consist of

- space translations (3 parameters)
- time translations (1 parameter)
- spatial rotations (3 parameters)
- (Galilean) boosts (3 parameters)
- discrete reflection symmetries

Noether's theorem: to each one-parameter group of continuous symmetries of a physical system corresponds a conserved quantity.

Galilean symmetries correspond to universal first integrals.

space translations	(linear) momentum
spatial rotations	angular momentum
time translations	energy
(Galilean) boosts	center of mass

The existence of universal symmetries is corroborated by the classical conservation laws (energy, momentum, etc.)!

Einstein: Galileo's relativity principle holds for all laws of physics (including those of electrodynamics), not only those of classical mechanics, but there is a limit speed c for the propagation of physical causes, i.e. there is no action at a distance.

Remark: the limit speed c equals the speed of light in vacuum.

Poincaré: the symmetry group of Maxwell's equations is the isometry group of 4D (homogeneous and isotropic) Minkowski space, the so-called Poincaré group $\mathcal{P}$, and not the Galilei group.

Remark: the Galilei group may be obtained from the Poincaré group by a limiting procedure (Wigner-Inönü contraction) when $c \to \infty$.

The true symmetry group of physical laws (at arbitrary speeds, but neglecting the effects of gravity) is the Poincaré group $\mathcal{P}$, containing

- space-time translations (4 parameters)
- space-time rotations, including the spatial rotations and the Lorentz-boosts (6 parameters)

Two relativistic first integrals: a 4D vector (four-momentum) and a 4D antisymmetric tensor (angular momentum).

In general relativity, flat Minkowski space-time is replaced by a curved manifold (a solution of Einstein's equations), and the Poincaré group by its isometry group (e.g. the de Sitter group).