Representation theory

1 Introduction

Structural study of a group can be quite involved if either the elements or the group multiplication are complicated.

Possible way out: consider another group whose algebraic structure is almost the same (i.e. nearly isomorphic), but with well-known elements and group multiplication (e.g. cyclic or permutation groups).

But cyclic groups are Abelian, while permutation groups are finite, hence one needs to consider a more general class: linear and unitary groups.

2 Linear and unitary groups

Let V denote a linear space with field of scalars $\mathbb{F}$ (usually either the real or the complex numbers). The general linear group

$$\mathsf{GL}(V) = \{A : V \to V \mid \det A \neq 0\}$$

over V consist of all invertible linear operators on V, with product the composition (multiplication) of operators.

A linear group is a subgroup of $\mathsf{GL}(V)$.

Remark. A subgroup of a linear group is itself linear.

The dimension $\dim G$ of a linear group $G < \mathsf{GL}(V)$ is the dimension (cardinality of any basis) of the linear space V.

Examples:

1. the group
$$\mathtt{Sc}(V) = \{\lambda \mathbf{id}_V \mid \lambda \in \mathbb{F}^\times\}$$
of scalar operators (isomorphic to the unit group $\mathbb{F}^\times$);

2. the special linear group
$$\mathsf{SL}(V) = \{A \in \mathsf{GL}(V) \mid \det A = 1\}$$
of unimodular operators (having determinant 1);

3. the stabilizer subgroup
$$\mathtt{Stab}(v) = \{A \in \mathsf{GL}(V) \mid Av = v\}$$
of some vector $v \in V$.

Linear groups are more amenable to study because they allow the use of

- linear algebra methods (e.g. spectral decomposition, determinants, etc.) in the study of their elements;
- constructions specific to linear groups (e.g. duality, direct sums and tensor products);
- special algorithms applicable to collections of linear operators.

Linear groups are rather the exception than the rule, but some special classes of groups (like finite, compact Lie, etc.) may be shown to have all members isomorphic to some linear group.

A complex inner product space is a complex linear space endowed with a map $\langle , \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that is

- linear in its first argument, i.e. $\langle \lambda a + \mu b, c \rangle = \lambda \langle a, c \rangle + \mu \langle b, c \rangle$ for all $\lambda, \mu \in \mathbb{C}$ and $a, b, c \in \mathcal{H}$;
- conjugate symmetric, i.e. $\langle b, a \rangle = \overline{\langle a, b \rangle}$ for all $a, b \in \mathcal{H}$, where the bar denotes complex conjugation;
- positive definite, i.e. $\langle a, a \rangle > 0$ for every non-zero element of $\mathcal{H}$.

Example: the space $\mathbb{C}^n$ of n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers with the inner product

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

A Hilbert space $\mathcal{H}$ is a complex inner product space that is complete (as a metric space) with respect to the norm topology induced by the metric

$$d(a,b) = \sqrt{\langle a-b, a-b\rangle}$$

A unitary operator on the Hilbert-space $\mathcal{H}$ is a (bounded) linear operator $U:\mathcal{H}\to\mathcal{H}$ preserving the inner product, i.e.

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

for $x, y\in\mathcal{H}$, while an antiunitary operator $A:\mathcal{H}\to\mathcal{H}$ is antilinear, i.e.

$$A\,(\alpha x+\beta y) = \overline{\alpha} Ax + \overline{\beta} Ay$$

for all $\alpha, \beta\in\mathbb{C}$ and $x, y\in\mathcal{H}$, and satisfies

$$\langle Ax, Ay\rangle = \overline{\langle x, y\rangle} = \langle y, x\rangle$$

Unitary operators are invertible, and the product of unitary operators is unitary, hence they form a group $\mathsf{U}(\mathcal{H})$ (denoted as $\mathsf{U}(n)$ in case $\mathcal{H}$ is of finite dimension n). A unitary group is any subgroup $G < \mathsf{U}(\mathcal{H})$.

The inverse of an antiunitary operator is antiunitary, while the product of two antiunitary operators is unitary; as a consequence, the collection $\widehat{\mathsf{U}(\mathcal{H})}$ of all unitary or antiunitary operators is itself a group, having $\mathsf{U}(\mathcal{H})$ as a subgroup of index 2.

Consequence: antiunitary operators form a coset of $\mathsf{U}(\mathcal{H})$, hence it is enough to understand the structure of the latter

$$\widehat{\mathsf{U}(\mathcal{H})} = \mathsf{U}(\mathcal{H}) \bigcup K\mathsf{U}(\mathcal{H})$$

for some fixed antiunitary K (chosen at will).

Unitary groups are easier to study because

1. to each linear subspace of a Hilbert space one can associate its so-called orthogonal complement;

2. the metric structure of a Hilbert space allows to single out especially useful bases (so-called orthonormal bases $\mathbf{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ that satisfy $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$);

3. any unitary operator can be diagonalized, and its eigenvalues lie on the complex unit circle $\mathsf{U}(1) = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$.

A group $G < \mathsf{GL}(V)$ is unitarizable, if V may be endowed with an inner product making it a Hilbert space $\mathcal{H}$ in such a way that $G < \mathsf{U}(\mathcal{H})$.

Unitary groups in physics: quantum systems and gauge symmetries.

(pure) states $\leftrightsquigarrow$ rays (1D subspaces) of Hilbert space
observable quantities $\leftrightsquigarrow$ self-adjoint operators on Hilbert space
energy $\leftrightsquigarrow$ Hamiltonian (generator of time translations)

Wigner's theorem: the symmetries of a quantum system correspond to (anti-)unitary operators commuting with its Hamiltonian.

Remark. Antiunitary symmetry operators are related to time reversal symmetry (exchanges cause and effect).

Gauge symmetries of fundamental interactions are described by (special) unitary groups in the Standard Model.

3 Matrix representations

If V has finite dimension n, then for any choice of basis $\mathbf{B}=\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ there exist a (non-canonical) isomorphism $\mathbf{\Gamma}_{\mathbf{B}}:\mathsf{GL}(V)\to\mathsf{GL}_n(\mathbb{F})$

$$A(\mathbf{e}_i)=\sum_{j=1}^{n}\mathbf{\Gamma}_{\mathbf{B}}(A)_{ij}\,\mathbf{e}_j$$

The matrix $\mathbf{\Gamma}_{\mathbf{B}}(A)\in\mathsf{GL}_n(\mathbb{F})$ associated to the linear operator $A\in\mathsf{GL}(V)$ with respect to the basis $\mathbf{B}=\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ provides a numerical representation that captures many algebraic properties, but it is by no means unique: one and the same operator could be represented by different matrices (with respect to different bases), and one and the same matrix may represent (with respect to different bases) different operators.

If $\mathbf{B}' = \{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ is another basis of V, then

$$\mathbf{\Gamma}_{\mathbf{B}'}(A) = C^{-1}\mathbf{\Gamma}_{\mathbf{B}}(A)\,C$$

where $\mathbf{e}'_i = \sum_j C_{ij}\mathbf{e}_j$ (C is the matrix of basis change).

$$\boxed{\textbf{matrices} \neq \textbf{linear operators}}$$

The image $\mathbf{\Gamma}_{\mathbf{B}}(G)$ of a linear group $G < \mathsf{GL}(V)$ is a matrix group.

If the Hilbert space $\mathcal{H}$ has finite dimension n, then for any choice of an orthonormal basis $\mathbf{B}$ the isomorphism $\mathbf{\Gamma}_{\mathbf{B}} \colon \mathsf{GL}(\mathcal{H}) \to \mathsf{GL}_n(\mathbb{C})$ assigns to each element of $\mathsf{U}(\mathcal{H})$ an n-by-n unitary matrix (recall that a matrix is unitary if its inverse equals the complex conjugate of its transpose).

4 Invariant subspaces and reducibility

A subset $W \subseteq V$ of a linear space V (with field of scalars $\mathbb{F}$) is a linear subspace if $x+y \in W$ and $\alpha x \in W$ whenever $x, y \in W$ and $\alpha \in \mathbb{F}$.

A linear subspace $W \subseteq V$ is nontrivial if it is different from the zero subspace and the whole space: in other words, if $0 < \dim W < \dim V$.

A linear subspace $W \subseteq V$ is an invariant subspace of the linear group $G < \mathsf{GL}(V)$ if all group elements map it onto itself, i.e. $gx \in W$ for all $g \in G$ and $x \in W$.

Remark. The zero subspace and the whole space are always invariant.

The translates $x+W=\{x+y \mid y\in W\}$ of a linear subspace $W\subseteq V$ (x runs over the elements of V) form themselves a linear space, the factor space V/W, with addition and multiplication by a scalar defined as

$$\begin{aligned}(x+W)+(y+W) &= (x+y)+W \\ \alpha(x+W) &= \alpha x+W\end{aligned}$$

for $x,y\in V$ and $\alpha\in\mathbb{F}$.

Remark. Any basis $\mathbf{B}_W$ of a linear subspace $W\subseteq V$ can be completed (in many different ways) to a basis $\mathbf{B}_V\supseteq\mathbf{B}_W$ of the whole space, and each element of the difference set $\mathbf{B}_V\backslash\mathbf{B}_W$ corresponds to a basis vector of the factor space.

Consequence: $\boxed{\dim(V/W)=\dim V-\dim W}$

Given an invariant subspace $W \subseteq V$ of $G < \mathsf{GL}(V)$, the restrictions

$$\begin{aligned} g_W : W &\to W \\ x &\mapsto gx \end{aligned}$$

to W of the group elements $g \in G$, and the factored operators

$$\begin{aligned} g/W : V/W &\to V/W \\ x + W &\mapsto gx + W \end{aligned}$$

are well defined linear operators making up linear groups

$$G_W = \{g_W \mid g \in G\} < \mathsf{GL}(W)$$

and

$$G/W = \{g/W \mid g \in G\} < \mathsf{GL}(V/W)$$

called respectively the reduction and factor of G.

Remark. Given a basis $\mathbf{B}_W$ of the invariant subspace $W < V$ and its completion $\mathbf{B}_V$ to a basis of V, the representation matrices with respect to $\mathbf{B}_V$ are upper triangular block matrices, i.e. they read

$$\mathbf{\Gamma}_{\mathbf{B_V}}(g) = \begin{pmatrix} \mathbf{\Gamma}_{\mathbf{B_W}}(g) & T(g) \\ 0 & \mathbf{\Gamma}_{\mathbf{B_V} \setminus \mathbf{B_W}}(g) \end{pmatrix}$$

for suitable matrices $T(g)$ that satisfy for all $g, h \in G$

$$T(gh) = \mathbf{\Gamma}_{\mathbf{B_W}}(g)\, T(h) + T(g)\, \mathbf{\Gamma}_{\mathbf{B_V} \setminus \mathbf{B_W}}(h)$$

A linear group is reducible or irreducible according to whether it has a nontrivial invariant subspace.

Schur's lemma: an operator that commutes with all elements of an irreducible linear group is a scalar multiple of the identity operator.

5 Linear representations

Problem: given an abstract group, find a linear group isomorphic to it.

There is usually no solution for a generic group, hence one needs to look for suitable (linear) homomorphic images.

A linear representation of the group G over the linear space V is a homomorphism $D : G \to \mathsf{GL}(V)$ into the general linear group over V.

The representation is called faithful if its kernel is trivial.

The image of a representation is always a linear group that is isomorphic to G precisely when the representation is faithful.

Remark. A linear representation $D : G \to \mathsf{GL}(V)$ over a linear space of dimension n (with field of scalars $\mathbb{F}$) determines, for any choice of a basis $\mathbf{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of the linear space V, a degree n matrix representation $D_{\mathbf{B}} : G \to \mathsf{GL}_n(\mathbb{F})$ via the rule $D_{\mathbf{B}} = \mathbf{\Gamma}_{\mathbf{B}} \circ D$, where $\mathbf{\Gamma}_{\mathbf{B}} : \mathsf{GL}(V) \to \mathsf{GL}_n(\mathbb{F})$ is the isomorphism associated to the basis $\mathbf{B}$.

As the theory becomes pretty complicated over generic linear spaces, we shall concentrate on representations over finite dimensional complex linear spaces, the so-called complex (linear) representations.

Remark. Some applications (e.g. in crystallography) involve the consideration of representations over the real and/or rational number field.

Examples of representations:

1. For any group G and any linear space V, the map

$$\begin{aligned} \mathbf{1}_V : G &\to \mathsf{GL}(V) \\ g &\mapsto \mathbf{id}_V \end{aligned}$$

assigning to each group element $g \in G$ the identity operator $\mathbf{id}_V$ on V is a representation, the so-called trivial representation of G over V;

2. For a linear group $G < \mathsf{GL}(V)$, the inclusion map

$$\begin{aligned} D_G : G &\to \mathsf{GL}(V) \\ g &\mapsto g \end{aligned}$$

that assigns to each group element $g \in G$ itself, is a representation of G, the so-called defining representation of G;

3. For a field $\mathbb{F}$ and a set X with n elements, the collection $\mathbb{F}(X)$ of all $\mathbb{F}$-valued functions on X is a linear space over $\mathbb{F}$ of dimension n (with pointwise addition and multiplication by elements of $\mathbb{F}$), and to each permutation $\pi \in \mathbb{S}_n$ the map

$$\begin{aligned} D(\pi) : \mathbb{F}(X) &\to \mathbb{F}(X) \\ f \quad &\mapsto f \circ \pi^{-1} \end{aligned}$$

assigns a linear operator acting on $\mathbb{F}(X)$ such that $D(\pi_1\pi_2) = D(\pi_1)\, D(\pi_2)$. Consequently, the map

$$\begin{aligned} D : G &\to \mathsf{GL}(\mathbb{F}(X)) \\ \pi &\mapsto D(\pi) \end{aligned}$$

is a representation of any subgroup $G < \mathsf{Sym}(X)$ over $\mathbb{F}(\mathbb{X})$, the so-called permutation representation associated to G.

A unitary representation of G on a Hilbert-space $\mathcal{H}$ is a homomorphism $U: G \to \mathsf{U}(\mathcal{H})$ into the group of unitary operators on $\mathcal{H}$, i.e. a linear representation all of whose representation operators are unitary:

$$\langle U(g)x, U(g)y \rangle = \langle x, y \rangle$$

for all $x, y \in \mathcal{H}$ and $g \in G$.

A unitarizable representation $D: G \to \mathsf{GL}(V)$ is one for which there exists a positive definite scalar product on V (making it a Hilbert-space) for which all representation operators are unitary.

All representations of a finite or compact group are unitarizable.

6 Equivalence and reducibility

The representations $D_1 : G \to \mathsf{GL}(V_1)$ and $D_2 : G \to \mathsf{GL}(V_2)$ are (linearly) equivalent, denoted $D_1 \cong D_2$, if there exists an invertible linear map (intertwiner) $A : V_1 \to V_2$ such that $D_2(g)A = AD_1(g)$ for all $g \in G$.

Remark. The dimension of equivalent representations is the same.

Linear equivalence is a reflexive, symmetric and transitive relation.

Equivalent representations are practically the same, e.g. the representation matrices of group elements coincide (w.r.t. suitable bases).

Problem: classify (up to equivalence) all representations of a given group.

A representation $D : G \to \mathsf{GL}(V)$ is called reducible if its image is a reducible linear group (i.e. $D(G) < \mathsf{GL}(V)$ has a nontrivial invariant subspace $W < V$), otherwise it is called irreducible.

Remark. One dimensional representations are always irreducible.

Schur's lemma: any operator that commutes with all representation operators of an irreducible representation is a multiple of the identity.

Remark. Since the symmetries of a quantum system are described by (anti-)unitary operators commuting with its Hamiltonian (by Wigner's theorem), the degeneracy of energy levels can be related to symmetries via Schur's lemma.

7 Direct sum of representations

The direct sum of the linear spaces V_1 and V_2 is the linear space $V_1 \oplus V_2$ whose elements are ordered pairs (x_1, x_2) with $x_1 \in V_1$ and $x_2 \in V_2$, while addition and multiplication by a scalar are defined component-wise:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

for $\alpha \in \mathbb{F}$ and $x_i, y_i \in V_i$ $(i=1,2)$.

Remark. If $\mathbf{B}_i$ denotes a basis of V_i, then a basis of $V_1 \oplus V_2$ is given by

$$\mathbf{B}_1 \sqcup \mathbf{B}_2 = \{(x_1, 0) \mid x_1 \in \mathbf{B}_1\} \cup \{(0, x_2) \mid x_2 \in \mathbf{B}_2\}$$

hence

$$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$$

The direct sum of $A_1 : V_1 \to V_1$ and $A_2 : V_2 \to V_2$ is the linear map

$$\begin{aligned} A_1 \oplus A_2 : V_1 \oplus V_2 &\to V_1 \oplus V_2 \\ (x_1, x_2) &\mapsto (A_1 x_1, A_2 x_2) \end{aligned}$$

The matrix of $A_1 \oplus A_2$ with respect to the basis $\mathbf{B}_1 \sqcup \mathbf{B}_2$ is block-diagonal

$$\mathbf{\Gamma}_{\mathbf{B_1} \sqcup \mathbf{B_2}}(A_1 \oplus A_2) = \begin{pmatrix} \mathbf{\Gamma}_{\mathbf{B_1}}(A_1) & 0 \\ 0 & \mathbf{\Gamma}_{\mathbf{B_2}}(A_2) \end{pmatrix}$$

Given representations $D_1 : G \to \mathsf{GL}(V_1)$ and $D_2 : G \to \mathsf{GL}(V_2)$, the map

$$\begin{aligned} D_1 \oplus D_2 : G &\to \mathsf{GL}(V_1 \oplus V_2) \\ g &\mapsto D_1(g) \oplus D_2(g) \end{aligned}$$

is a new representation, the direct sum of D_1 and D_2, whose equivalence class is completely determined by the classes of D_1 and D_2.

The direct sum is commutative and associative (up to equivalence)

$$D_1 \oplus D_2 \cong D_2 \oplus D_1$$
$$D_1 \oplus (D_2 \oplus D_3) \cong (D_1 \oplus D_2) \oplus D_3$$

Remark. The subspaces $\hat{V}_1 = \{(x_1, 0) \mid x_1 \in V_1\}$ and $\hat{V}_2 = \{(0, x_2) \mid x_2 \in V_2\}$ are invariant subspaces of the direct sum $D_1 \oplus D_2$, with corresponding reductions to $\hat{V}_i$ equivalent to D_i for $i = 1, 2$, i.e.

$$(D_1 \oplus D_2)_{\hat{V}_i} \cong D_i$$

A representation is completely reducible if it can be decomposed into a direct sum of irreducible representations (e.g. trivial representations, which can be decomposed into a direct sum of trivial representations over one dimensional subspaces).

Completely reducible representations have an irreducible decomposition

$$\bigoplus_{\mathbf{i}\in\mathsf{Irr}(G)} n_{\mathbf{i}}\mathbf{i}$$

into a direct sum of irreducibles, where $n_{\mathbf{i}}\in\mathbb{Z}_+$ is the multiplicity of the irreducible $\mathbf{i}\in\mathsf{Irr}(G)$.

completely reducible representations $\leftrightsquigarrow$ maps from $\mathsf{Irr}(G)$ into $\mathbb{Z}_+$

Maschke's theorem: all complex representations of a finite group are completely reducible (a similar result, the Peter-Weyl theorem, holds for the representations of compact topological groups).

More generally, all unitary representations are completely reducible.

8 The contragredient

The dual $V^\vee$ of a linear space V (with field of scalars $\mathbb{F}$) is the set of all linear functionals, i.e. maps $\phi : V \to \mathbb{F}$ from V into $\mathbb{F}$ such that

$$\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$$

for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in V$. The dual is itself a linear space over $\mathbb{F}$ with pointwise operations

$$\begin{aligned} \phi_1 + \phi_2 : V &\to \mathbb{F} \\ x &\mapsto \phi_1(x) + \phi_2(x) \end{aligned}$$

and

$$\begin{aligned} \alpha\phi_1 : V &\to \mathbb{F} \\ x &\mapsto \alpha\phi_1(x) \end{aligned}$$

for $\alpha \in \mathbb{F}$ and $\phi_1, \phi_2 \in V^\vee$.

If $\mathbf{B} = \{b_1, \ldots, b_n\}$ is a basis of V, then the functionals

$$\begin{aligned} b_i^\vee : V &\to \mathbb{F} \\ b_j &\mapsto \delta_{ij} \end{aligned}$$

form a basis of $V^\vee$, the dual basis $\mathbf{B}^\vee = \{b_i^\vee \mid b_i \in \mathbf{B}\}$.

Consequence: $\dim V^\vee = \dim V$.

To each $v \in V$ one can associate a linear functional

$$\begin{aligned} v^\flat : V^\vee &\to \mathbb{F} \\ \phi &\mapsto \phi(v) \end{aligned}$$

of the dual space $V^\vee$, and the mapping

$$\begin{aligned} \flat : V &\to (V^\vee)^\vee \\ v &\mapsto v^\flat \end{aligned}$$

is an invertible linear map.

$\boxed{\text{The dual of the dual can be identified naturally with the original space.}}$

The transpose $A^{\mathtt{tr}}$ of a linear map $A\colon V_1 \to V_2$ is the linear map

$$\begin{aligned} A^{\mathtt{tr}} : V_2^\vee &\to V_1^\vee \\ \phi &\mapsto \phi \circ A \end{aligned}$$

between dual spaces.

Given a basis $\mathbf{B} = \{b_1, \ldots, b_n\}$ of V, the matrix of the transpose of $A \in \mathsf{GL}(V)$ with respect to the dual basis $\mathbf{B}^\vee = \{b_i^\vee \mid b_i \in \mathbf{B}\}$ reads

$$\boldsymbol{\Gamma}_{\mathbf{B}^\vee}(A^{\mathtt{tr}}) = \boldsymbol{\Gamma}_{\mathbf{B}}(A)^{\intercal}$$

where $M^{\intercal}$ denotes the transpose of $M \in \mathrm{Mat}_n(\mathbb{F})$, i.e. $M^{\intercal}_{ij} = M_{ji}$.

Transposition is an antihomomorphism, i.e. $\boxed{(AB)^{\mathtt{tr}} = B^{\mathtt{tr}} A^{\mathtt{tr}}}$.

For a representation $D\colon G \to \mathsf{GL}(V)$ over the linear space V, the mapping

$$\begin{aligned} D^\vee : G &\to \mathsf{GL}(V^\vee) \\ g &\mapsto D^{\mathtt{tr}}\left(g^{-1}\right) \end{aligned}$$

which assigns to each group element the transpose of the representation operator of its inverse is a representation of G on the dual space $V^\vee$, the contragredient of D (note that $\dim D^\vee = \dim D$).

The contragredient of the contragredient is the original representation

$$(D^\vee)^\vee \cong D$$

and the contragredient of a direct sum is the sum of the contragredients

$$\Big(\bigoplus_i D_i\Big)^\vee \cong \bigoplus_i D_i^\vee$$

The contragredient of an irreducible representation is itself irreducible.

9 Tensor products and the fusion ring

A bilinear functional on the linear spaces V_1 and V_2 (with common field of scalars $\mathbb{F}$) is a map $\mathfrak{b}: V_1 \times V_2 \to \mathbb{F}$ that is linear in each of its arguments:

$$\begin{aligned}\mathfrak{b}(\alpha x_1 + \beta y_1, x_2) &= \alpha \mathfrak{b}(x_1, x_2) + \beta \mathfrak{b}(y_1, x_2) \\ \mathfrak{b}(x_1, \alpha x_2 + \beta y_2) &= \alpha \mathfrak{b}(x_1, x_2) + \beta \mathfrak{b}(x_1, y_2)\end{aligned}$$

for scalars $\alpha, \beta \in \mathbb{F}$ and $x_i, y_i \in V_i$ $(i = 1, 2)$.

The set $\mathcal{B}(V_1, V_2)$ of bilinear functionals is a linear space with the obvious pointwise operations

$$\begin{aligned}\mathfrak{b}_1 + \mathfrak{b}_2 : V_1 \times V_2 &\to \mathbb{F} \\ (v, w) &\mapsto \mathfrak{b}_1(v, w) + \mathfrak{b}_2(v, w)\end{aligned}$$

and

$$\begin{aligned} \alpha\mathfrak{b} : V_1 \times V_2 &\to \mathbb{F} \\ (v, w) &\mapsto \alpha\mathfrak{b}(v, w) \end{aligned}$$

for $\alpha \in \mathbb{F}$ and $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{B}(V_1, V_2)$.

The dyadic product $v_1 \otimes v_2$ of $v_1 \in V_1$ and $v_2 \in V_2$ is the linear functional

$$\begin{aligned} v_1 \otimes v_2 : \mathcal{B}(V_1, V_2) &\to \mathbb{F} \\ \mathfrak{b} &\mapsto \mathfrak{b}(v_1, v_2) \end{aligned}$$

i.e. evaluation of bilinear functionals at the given arguments.

Dyadic products span the dual of the space of bilinear functionals,

called the tensor product $V_1 \otimes V_2$ of the linear spaces V_1 and V_2

$$\boxed{V_1 \otimes V_2 = \mathcal{B}(V_1, V_2)^\vee}$$

Physical significance: the state space of a composite quantum system is the tensor product of the Hilbert spaces of its components; dyadic products correspond to separable states, with no correlation between the subsytems.

Given bases $\mathbf{B}_i$ of V_i $(i=1,2)$, the set $\{e \otimes f \mid e \in \mathbf{B}_1, f \in \mathbf{B}_2\}$ of dyadic products is a basis (the product basis) of the tensor product $V_1 \otimes V_2$.

Consequence: $\boxed{\dim(V_1 \otimes V_2) = \dim V_1 \dim V_2}$

Remark. While every element of $V_1 \otimes V_2$ is a linear combination of dyadic products, not all of them are dyadic products themselves (corresponding to inseparable states in quantum theory, leading to the phenomenon of quantum entanglement).

The tensor product of the linear operators $A_1 : V_1 \to W_1$ and $A_2 : V_2 \to W_2$ is the linear operator $A_1 \otimes A_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ that maps each dyadic product $v_1 \otimes v_2$ into $A_1 v_1 \otimes A_2 v_2$ (well-defined, since dyadic products span the tensor product space).

Given operators $A_i \in \mathsf{GL}(V_i)$ and bases $\mathbf{B}_i$ of V_i $(i = 1, 2)$, the matrix of the tensor product $A_1 \otimes A_2 \in \mathsf{GL}(V_1 \otimes V_2)$ with respect to the product basis $\{e \otimes f \mid e \in \mathbf{B}_1, f \in \mathbf{B}_2\}$ is the Kronecker product of the matrices $\mathbf{\Gamma}_{\mathbf{B_1}}(A_1)$ and $\mathbf{\Gamma}_{\mathbf{B_2}}(A_2)$, with matrix elements

$$[\mathbf{\Gamma}_{\mathbf{B_1}}(A_1) \otimes \mathbf{\Gamma}_{\mathbf{B_2}}(A_2)]_{(ip)(jq)} = [\mathbf{\Gamma}_{\mathbf{B_1}}(A_1)]_{ij} \, [\mathbf{\Gamma}_{\mathbf{B_2}}(A_2)]_{pq}$$

for $i, j \in \mathbf{B}_1$ and $p, q \in \mathbf{B}_2$.

Remark. The trace of a tensor product is the product of the traces

$$\mathrm{Tr}\,\{A_1 \otimes A_2\} = \mathrm{Tr}\,\{A_1\}\,\mathrm{Tr}\,\{A_2\}$$

The tensor product of $D_1 : G \to \mathsf{GL}(V_1)$ and $D_2 : G \to \mathsf{GL}(V_2)$ is the representation

$$\begin{aligned} D_1 \otimes D_2 : G &\to \mathsf{GL}(V_1 \otimes V_2) \\ g &\mapsto D_1(g) \otimes D_2(g) \end{aligned}$$

The tensor product is compatible with linear equivalence, i.e. the class of a product is determined by the classes of its factors. Moreover, it is commutative and associative (up to equivalence),

$$D_1 \otimes D_2 \cong D_2 \otimes D_1 \quad \text{és} \quad D_1 \otimes (D_2 \otimes D_3) \cong (D_1 \otimes D_2) \otimes D_3$$

and distributive with respect to direct sums

$$D_1 \otimes (D_2 \oplus D_3) \cong (D_1 \otimes D_2) \oplus (D_1 \otimes D_3)$$

The identity representation $\mathbf{1}$, which is (the equivalence class of) any 1D trivial representation, is an identity element for the tensor product:

$$\boxed{\mathbf{1} \otimes D \cong D \otimes \mathbf{1} \cong D}$$

In general, the tensor product of any irreducible with a representation of dimension 1 is itself irreducible.

Remark. The tensor product of two irreducibles may contain the identity at most once, in which case they are contragredients of each other.

Equivalence classes of representations with the operations of direct sum and tensor product form the fusion ring of the group under study (which is actually not a ring, since representations have no additive inverses).

For completely reducible representations (e.g. finite or compact groups) it is enough to know the fusion rules

$$\mathfrak{i} \otimes \mathfrak{j} = \bigoplus_{\mathfrak{p} \in \mathsf{Irr}(G)} N_{\mathfrak{i}\mathfrak{j}}^{\mathfrak{p}} \mathfrak{p}$$

i.e. the irreducible decomposition of the tensor products of irreducibles (where $\mathfrak{i}, \mathfrak{j} \in \mathsf{Irr}(G)$ denote irreducibles of G).

The non-negative integer multiplicities $N_{\mathfrak{i}\mathfrak{j}}^{\mathfrak{p}} \in \mathbb{Z}_+$ are the so-called fusion coefficients of the group.

Physics relevance: composition of symmetry charges (e.g. angular momentum of composite systems).

10 Representation characters

Basic tasks of representation theory:

1. Classify all representations of a given group (up to equivalence). This involves in particular deciding whether two representations are equivalent, and (for completely reducible representations) compute irreducible decompositions.

2. Determine the fusion and branching rules, symmetric powers, etc.

3. Use all the above to get structural information on the given group.

Question: is there a simple numerical characterization of the equivalence classes of representations.

Frobenius' observation: for a representation $D : G \rightarrow \mathsf{GL}(V)$ over the complex numbers, the function

$$\begin{aligned} \chi_D : G &\rightarrow \mathbb{C} \\ g &\mapsto \mathrm{Tr}\,\{D(g)\} \end{aligned}$$

that assigns to each group element the trace of its representation operator (called the character of the representation) provides such a tool.

Two linear representations are equivalent precisely when their characters coincide (especially useful for finite groups).

Remark. Since all elements of a finite group have finite order, all eigenvalues of the representation operators are roots of unity, hence

$$\begin{aligned}\chi_D\left(g^{-1}\right) &= \overline{\chi_D(g)} \\ |\chi_D(g)| &\leq \chi_D(1) = \dim D\end{aligned}$$

for representations of finite groups.

The character of the contragredient reads

$$\chi_{D^\vee}(g) = \chi_D\left(g^{-1}\right)$$

while those of direct sums and tensor products is given by

$$\begin{aligned}\chi_{D_1 \oplus D_2}(g) &= \chi_{D_1}(g) + \chi_{D_2}(g) \\ \chi_{D_1 \otimes D_2}(g) &= \chi_{D_1}(g)\,\chi_{D_2}(g)\end{aligned}$$

The character of a completely reducible representation D reads

$$\chi_D = \sum_{\mathbf{i}\in\mathsf{Irr}(G)} n_{\mathbf{i}}\chi_{\mathbf{i}}$$

with $n_{\mathbf{i}}$ denoting the multiplicity of the irreducible $\mathbf{i}\in\mathsf{Irr}(G)$ in the decomposition of D.

Representation characters are class functions, i.e. they are constant on conjugacy classes of elements:

$$\chi_D\left(h^{-1}gh\right)=\chi_D(g)$$

for all $g,h\in G$, because

$$\mathrm{Tr}\left\{D\left(h^{-1}gh\right)\right\}=\mathrm{Tr}\left\{D\left(h^{-1}\right)D(g)D(h)\right\}=\mathrm{Tr}\left\{D\left(hh^{-1}\right)D(g)\right\}=\mathrm{Tr}\left\{D(g)\right\}$$

Remark. Class functions form a linear space over $\mathbb{C}$.

In case the group G is finite, the space of class functions becomes a finite dimensional Hilbert-space with scalar product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \, \overline{\psi(g)}$$

Orthogonality relations: the irreducible characters of a finite group form an orthonormal basis in the space of class functions.

$$\langle \chi_{\mathbf{i}}, \chi_{\mathbf{j}} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathbf{i}}(g) \, \overline{\chi_{\mathbf{j}}(g)} = \delta_{\mathbf{ij}}$$

Remark. A similar result holds for compact groups, with summation replaced by the Haar-integral in the definition of the scalar product.

Consequences:

1. # of inequivalent irreducible representations = # of conjugacy classes

2.
$$\sum_{\mathfrak{i}\in \mathsf{Irr}(G)} (\dim \mathfrak{i})^2 = \sum_i \chi_{\mathfrak{i}}(1)^2 = |G| \qquad \text{Burnside's theorem}$$

3.
$$\boxed{n_{\mathfrak{i}} = \langle \chi_D , \chi_{\mathfrak{i}} \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_D(g)\, \overline{\chi_{\mathfrak{i}}(g)}}$$

is the multiplicity of an irreducible $\mathfrak{i}\in \mathsf{Irr}(G)$ in the decomposition of D.

In particular, the fusion coefficients are given by

$$N_{\mathfrak{i}\mathfrak{j}}^{\mathfrak{p}} = \langle \chi_{\mathfrak{i}\otimes\mathfrak{j}} , \chi_{\mathfrak{p}} \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_{\mathfrak{i}}(g)\, \chi_{\mathfrak{j}}(g)\, \overline{\chi_{\mathfrak{p}}(g)}$$

Knowledge of the characters solves most fundamental problems!

Character table of $\mathbb{D}_3$

	$\mathcal{C}_1 = \{1\}$	$\mathcal{C}_2 = \left\{C, C^{-1}\right\}$	$\mathcal{C}_3 = \{\sigma_1, \sigma_2, \sigma_3\}$
$\mathbf{1}$	1	1	1
$\mathbf{1}^*$	1	1	-1
$\mathbf{2}$	2	-1	0

Fusion rules of $\mathbb{D}_3$

	$\mathbf{1}$	$\mathbf{1}^*$	$\mathbf{2}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}^*$	$\mathbf{2}$
$\mathbf{1}^*$	$\mathbf{1}^*$	$\mathbf{1}$	$\mathbf{2}$
$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1} \oplus \mathbf{1}^* \oplus \mathbf{2}$

11 Symmetric powers

For any linear space V, the braiding operator $\mathcal{R}: V \otimes V \to V \otimes V$ that permutes the factors of dyadic products

$$\mathcal{R}(v_1 \otimes v_2) = v_2 \otimes v_1$$

is an involution (its square is the identity), hence its eigenvalues are ± 1.

As a consequence, one has the spectral decomposition

$$\boxed{V \otimes V = \wedge_+^2 V \oplus \wedge_-^2 V}$$

into eigenspaces of $\mathcal{R}$ (symmetric and antisymmetric subspaces).

If $\mathbf{B}=\{b_1,\ldots,b_n\}$ is a basis of V, then the sets

$$\wedge_+^2\mathbf{B} = \{b_i\otimes b_j + b_j\otimes b_i \,|\, 1\le i\le j\le n\}$$
$$\wedge_-^2\mathbf{B} = \{b_i\otimes b_j - b_j\otimes b_i \,|\, 1\le i< j\le n\}$$

are natural bases of $\wedge_\pm^2 V$, hence

$$\dim \wedge_\pm^2 V = \frac{n(n\pm 1)}{2}$$

For any operator $A\in\mathsf{GL}(V)$, its tensor square $A\otimes A$ commutes with the braiding operator $\mathcal{R}$, since dyadic products span the tensor product and

$$\{\mathcal{R}\circ(A\otimes A)\}(v_1\otimes v_2) = \mathcal{R}(Av_1\otimes Av_2) = Av_2\otimes Av_1 =$$
$$= (A\otimes A)(v_2\otimes v_1) = \{(A\otimes A)\circ\mathcal{R}\}(v_1\otimes v_2)$$

for any $v_1, v_2\in V$.

As a consequence, the eigenspaces $\wedge^2_\pm V$ of $\mathcal{R}$ are invariant subspaces of the operator $A \otimes A$, and the reduction of the tensor square to these subspaces are the symmetric and antisymmetric squares $\wedge^2_\pm A$ of A.

Remark.

$$\mathrm{Tr}\left\{\wedge^2_\pm A\right\} = \frac{\mathrm{Tr}\left\{A\right\}^2 \pm \mathrm{Tr}\left\{A^2\right\}}{2}$$

Since the representation operators of the tensor square $D \otimes D$ of any representation $D\colon G \to \mathsf{GL}(V)$ have the form $D(g) \otimes D(g)$, the maps

$$\begin{aligned} \wedge^2_\pm D : G &\to \mathsf{GL}\left(\wedge^2_\pm V\right) \\ g &\mapsto \wedge^2_\pm D(g) \end{aligned}$$

define representations $\wedge^2_\pm D$ of G, the symmetric and antisymmetric squares of D.

Physics relevance: second quantization of composite systems made up of identical subsystems. According to the Pauli principle, the physical states belong to either the symmetric or the antisymmetric powers (for bosonic, resp. fermionic excitations).

Remark. In space-time dimension $2 = 1+1$, the braiding of particles is no more involutive (because of the topology of the light cone), hence it may have eigenvalues differing from ± 1. This leads, e.g. in (quasi-)2D electronic systems relevant to the quantum Hall effect, to the possibility of anyonic excitations that obey so-called braid statistics, a signal of quantum symmetries that cannot be described via the group concept, but need a more general framework.

12 Branching rules

The restriction of a representation $D : G \to \mathsf{GL}(V)$ to a subgroup $H < G$ is the representation

$$\begin{aligned} \mathtt{res}_H D : H &\to \mathsf{GL}(V) \\ h &\mapsto D(h) \end{aligned}$$

of the subgroup H.

Restriction is compatible with direct sums and a tensor products

$$\begin{aligned} \mathtt{res}_H (D_1 \oplus D_2) &= \mathtt{res}_H D_1 \oplus \mathtt{res}_H D_2 \\ \mathtt{res}_H (D_1 \otimes D_2) &= \mathtt{res}_H D_1 \otimes \mathtt{res}_H D_2 \end{aligned}$$

One can restrict in stages: if $K < H < G$ and $D : G \to \mathsf{GL}(V)$, then

$$\mathtt{res}_K (\mathtt{res}_H D) = \mathtt{res}_K D$$

For completely reducible representations it is sufficient to know the so-called branching rules

$$\mathtt{res}_H \mathfrak{i} = \bigoplus_{\mathfrak{p} \in \mathsf{Irr}(H)} B_{\mathfrak{i}}^{\mathfrak{p}} \mathfrak{p}$$

(where the $B_{\mathfrak{i}}^{\mathfrak{p}}$ are non-negative integer multiplicities), the restrictions of the irreducible representations $\mathfrak{i} \in \mathsf{Irr}(G)$.

Branching rules are important in the description of the phenomenon of symmetry breaking, when the ground state of a system does not respect all the symmetries of the dynamics.

13 Projective representations

Because the (pure) states of a quantum system correspond to the one dimensional subspaces of its Hilbert–space $\mathcal{H}$, the action of symmetry operators is only determined up to scalar factors. As a consequence, the product of two symmetry operators is not necessarily a symmetry operator itself, but a scalar multiple of a symmetry operator.

If the action of the symmetry transformation $g \in G$ is described by the unitary operator $U(g)$ acting on the Hilbert–space, then

$$U(g)\, U(h) = \alpha(g,h)\, U(gh)$$

with some suitable scalar factors $\alpha(g,h) \in \mathbb{C}$.
Note that $U(\mathbf{1}_G) = \mathbf{id}_{\mathcal{H}}$ implies $\alpha(\mathbf{1}_G, g) = \alpha(g, \mathbf{1}_G) = 1$ for all $g \in G$.

In complete generality, a mapping $\mathfrak{D}: G \to \mathsf{GL}(V)$ for which $\mathfrak{D}(1) = \mathbf{id}_V$ is called a projective representation of the group G over the linear space V if there exists a function $\alpha : G \times G \to \mathbb{C}^{\times}$ (the cocycle of $\mathfrak{D}$) such that

$$\boxed{\mathfrak{D}(g)\,\mathfrak{D}(h) = \alpha(g,h)\,\mathfrak{D}(gh)}$$

for all $g, h \in G$.

Since the product of operators is associative, the cocycle satisfies

$$\boxed{\alpha(g_1,g_2)\alpha(g_1 g_2,g_3) = \alpha(g_1,g_2 g_3)\alpha(g_2,g_3)} \qquad \text{cocycle equation}$$

and is normalized, i.e. $\alpha(g,1) = \alpha(1,g) = 1$.

The pointwise product of two cocycles is again a cocycle, hence cocycles form an Abelian group, whose identity element is the trivial cocycle (with all values equal to 1).

Remark. Ordinary representations are the projective representations with trivial cocycle.

The projective representations $\mathfrak{D}_1 : G \to \mathsf{GL}(V_1)$ and $\mathfrak{D}_2 : G \to \mathsf{GL}(V_2)$ are projectively equivalent if there exists a bijective linear map $A : V_1 \to V_2$ and a function $\lambda : G \to \mathbb{C}^\times$ (the scale factor) such that for all $g \in G$

$$\mathfrak{D}_2(g)A = \lambda(g)\, A\mathfrak{D}_1(g)$$

Cocycles of equivalent projective representations are cohomologous, i.e. there exists a function $\lambda : G \to \mathbb{C}^\times$ such that

$$\frac{\alpha_2(g,h)}{\alpha_1(g,h)} = \frac{\lambda(g)\,\lambda(h)}{\lambda(gh)}$$

for all $g, h \in G$.

Being cohomologous is a congruence relation of the group of cocycles, hence its equivalence classes form a group, the Schur multiplier of G.

The Schur multiplier of a finite or compact group (and of most groups of practical interest) is finite, but it can be infinite for some important groups (e.g. for the 2D conformal group, a fact of extreme importance in string theory).

Remark. Higher cohomology groups can be defined analogously (with 2-cocycles replaced by scalar valued functions on G^n that satisfy a suitable cocycle equation). These form the basis of the cohomology theory of groups, and find interesting physics applications, e.g. in the study of gauge and gravitational anomalies, Chern-Simons dynamics,etc.

$\hat{G}$ is called a covering group of G if it has a central subgroup $A < Z(\hat{G})$ isomorphic with the Schur-multiplier of G such that the corresponding factor group is isomorphic with G itself

$$\hat{G}/A \cong G$$

There is a one-to-one correspondence between projective representations of a group G and ordinary representations of its covering group $\hat{G}$, with irreducibles corresponding to irreducibles.

Schur: every finite group has a covering group (but there might be several non-isomorphic ones)!

Examples:

1. $\hat{\mathbf{T}} \cong \mathsf{SL}(2,\mathbb{Z}_3)$, the binary tetrahedral group;

2. $\hat{\mathbf{I}} \cong \mathsf{SL}(2,\mathbb{Z}_5)$, the binary icosahedral group;

3. $\widehat{\mathbb{A}_6} \cong 3{\cdot}\mathsf{SL}(2,\mathbb{Z}_9)$, due to the exceptional outer automorphism;

For Lie-groups, the covering group of G is the unique simply connected group $\hat{G}$ locally isomorphic to G (the universal cover).

In particular, since $\mathsf{SU}(2)$ is simply connected and locally isomorphic to $\mathsf{SO}(3) \cong \mathsf{SU}(2)/\mathsf{Z}(\mathsf{SU}(2))$, the Schur multiplier of $\mathsf{SO}(3)$ is $\mathbb{Z}_2$, leading to two classes of projective representations for the rotation group: tensorial (ordinary) and spinorial ones.