Simple models with Alice fluxes

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Abstract

We introduce two simple models which feature an Alice electrodynamics phase. In a well defined sense the Alice flux solutions we obtain in these models obey first order equations similar to those of the Nielsen-Olesen fluxtube [1] in the abelian higgs model in the Bogomol'nyi limit. Some numerical solutions are presented as well.

1 Alice electrodynamics

Alice Electrodynamics is a theory with gauge group $H = U(1) \ltimes \mathbb{Z}_2$, i.e. an absolute minimally non-abelian extension of ordinary electrodynamics, where charge conjugation has been turned into a local symmetry. As this non-abelian extension is discrete, it only effects 'electrodynamics' through certain global (topological) features, such as the existence of Alice fluxes and Cheshire charges [2, 3]. In an Alice phase of some theory one has the possibility of a topological stable flux tube or string coexisting with an unbroken $U(1)$, because the connectivity of H is nontrivial: $\Pi_0(H) = \mathbb{Z}_2$. The Alice phase is usually obtained by spontaneous breaking of a larger, continuous non-abelian symmetry group. The original Alice model studied by Schwarz [2], is a $SU(2)$ theory spontaneously broken down to a $U(1) \ltimes \mathbb{Z}_2$ by a higgs field in the 5−dimensional representation of (see also[4]). The higgs field is chosen in this representation, because it is the smallest irreducible representation which admits H as a residual symmetry group and allows for a single valued vacuum configuration that supports Alice fluxes.

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In this paper we will discuss two alternative models, which support an Alice phase. Before doing so, we briefly review the salient features of the model discussed in [2, 4].
The action is given by:

$$S = \int d^4x \; \{Tr\; \{-\frac{1}{8}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}D^{\mu}\Phi D_{\mu}\Phi\; \} + V(\Phi)\} \quad , \tag{1}$$

where the higgs field $\Phi = \Phi^{ab}$ is a real symmetric traceless 3×3 matrix.
The most general renormalisable potential is given by:

$$V = -\frac{1}{2}\mu^2 Tr\Phi^2 - \frac{1}{3}\gamma Tr\Phi^3 + \frac{1}{4}\lambda(Tr\Phi^2)^2 \tag{2}$$

By a suitable choice of parameters the higgs field will acquire a vacuum expectation value, Φ_0. In a gauge where Φ_0 is diagonal it takes the form $\Phi_0 = diag(a, b, -a-b)$. For a certain range of potential parameters one furthermore has that $a = b$, so that Φ_0 is given by:

$$\Phi_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \tag{3}$$

with $a = b = \frac{\gamma \pm \sqrt{\gamma^2 + 24\mu^2\lambda}}{12\lambda}$. Indeed, this ground state is invariant under rotations around the T_3-axis ($U(1)$) and invariant under rotations by an angle π around any axis perpendicular to the T_3-direction ($\mathbb{Z}_2$). These two transformations do not commute with each other, in fact they anti-commute, so the resulting residual gauge group is indeed $U(1) \ltimes \mathbb{Z}_2$. This means that we have Alice electrodynamics as the low energy effective theory in this model.
An alternative way to see the structure of the residual gauge group, is to think of the higgs field as the symmetric traceless product of two vectors, ϕ_1 and ϕ_2,

$$\Phi^{ab} = \phi_1^a\phi_2^b + \phi_2^a\phi_1^b - \frac{2}{3}\delta^{ab}(\vec{\phi_1} \cdot \vec{\phi_2}). \tag{4}$$

If both isovectors, $\vec{\phi_i}$, are non zero, there is in general only a $\mathbb{Z}_2$ gauge symmetry left, $\vec{\phi_i} \rightarrow \vec{\phi_i'} = -\vec{\phi_i}$. However, in case that both isovectors are (anti-)parallel, the gauge group is $U(1) \ltimes \mathbb{Z}_2$. If one of the isovectors is zero, the gauge group is not broken at all and the symmetry remains $SU(2)$. These are the residual gauge groups which one may encounter in this model. It is easy to show that the case where the two isovectors are (anti-)parallel, corresponds to the situation where $\Phi = \Phi_0$.

1.1 The Alice flux solution

In this section we will present explicit regular solutions, corresponding to an Alice flux tube along the z-axis, which where constructed in [5].
To have a static finite energy solution, all terms in the energy density should go to zero at spatial infinity. Thus the covariant derivatives need to vanish at spatial infinity. Let's look at the angular derivative, the condition $D_\theta\Phi = 0$ tells us that the higgs field has the following form at spatial infinity.

$$\Phi(\theta) = S(\theta)\Phi(0)S^{-1}(\theta) \tag{5}$$

with

$$S(\theta)=\exp\{e\int_0^\theta rA_\theta d\theta\} \tag{6}$$

Since we are looking for solutions which correspond to an Alice flux, $S(2\pi)$ needs to be an element of the disconnected part of the (residual) gauge group. A simple choice for A_θ doing this is $A_\theta=\frac{1}{2er}T_1$.
This leads to the ansatz:

$$\begin{aligned} A_\theta &= \frac{\alpha(r)}{2er}T_1 && (7)\\ \Phi(r,\theta) &= e^{\frac{\theta T_1}{2}}\Phi(r)e^{-\frac{\theta T_1}{2}} && (8)\end{aligned}$$

where the tensor $\Phi(r)$ is conveniently parameterized as,

$$\Phi(r)=m(r)\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&0\\0&0&-\frac{1}{2}\end{pmatrix}+q(r)\begin{pmatrix}0&0&0\\0&\frac{3}{2}&0\\0&0&-\frac{3}{2}\end{pmatrix} \tag{9}$$

The part proportional to $m(r)$ is the part of the higgs field that is invariant under rotations generated by T_1. The boundary condition at spatial infinity is $m(\infty)=q(\infty)$, implying that $\Phi(\infty)$ is of the form (3), i.e. the residual symmetry is $U(1)\ltimes\mathbb{Z}_2$ indeed, where the electrodynamic $U(1)$ is generated by T_3. At the origin, m and q have to satisfy different boundary conditions; the field $q(r)$ needs to go to zero. The term proportional to $m(r)$ is invariant under T_1 rotations, therefore $m(r)$ does not need to go to zero. Again, this means that the higgs field is of the form (3), i.e. the unbroken gauge group is $U(1)\ltimes\mathbb{Z}_2$. However, the unbroken $U(1)$ is generated by T_1. Finally, the field $\alpha(r)$ needs to be zero at the origin and unity at spatial infinity.
Inserting this ansatz in the field equations gives, after suitable rescalings, the following set of equations.

$$\begin{aligned}\partial_r^2\alpha(r)-\frac{1}{r}\partial_r\alpha(r) &= 9q^2(r)(\alpha(r)-1) && (10)\\ \partial_r^2 q(r)+\frac{1}{r}\partial_r q(r) &= \frac{(\alpha(r)-1)^2q(r)}{r^2}+\xi(9q^2(r)+3m^2(r)-2)q(r)\\ &\quad +2\chi m(r)q(r) && (11)\\ \partial_r^2 m(r)+\frac{1}{r}\partial_r m(r) &= \xi(9q^2(r)+3m^2(r)-2)m(r)+\chi(3q^2(r)-m^2(r)) && (12)\end{aligned}$$

We summarize the boundary values for the rescaled fields below:

field	$r\to 0$	r$\to\infty$
$\alpha(r)$	0	1
$q(r)$	0	$q(\infty)$
$m(r)$	*constant*	$m(\infty)$

where,

$$m(\infty)=q(\infty)=\frac{-\chi\pm\sqrt{\chi^2+24\xi^2}}{12\xi}\equiv a(\xi,\chi) \tag{13}$$

with $\xi = \frac{\lambda}{e^2}$ and $\chi = \frac{\gamma\sqrt{\lambda}}{\mu e^2}$.
The system (10 - 12) was solved numerically with the help of a relaxation method in [5]. The solution for the potential parameter values $\xi = 1$ and $\chi = -1$ is given in figure 1 . The situation at hand is reminiscent to the one considered by Witten [6] for a $U(1) \times \tilde{U}(1)$ model, in the sense that we have an unbroken $U(1) \ltimes \mathbb{Z}_2$ at the core and a different $U(1) \ltimes \mathbb{Z}_2$ at infinity. However, the crucial difference is, that our 'two' $U(1)$ gauge groups do not commute with each other.

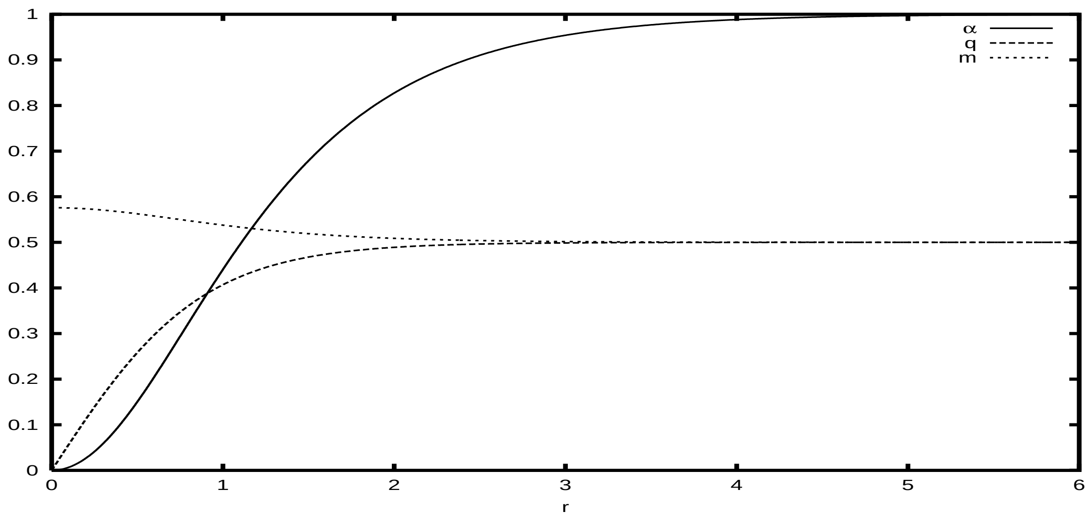

Figure 1: A regular solution for the fields with an Alice flux for $\xi = 1$ and $\chi = -1$.

Interestingly, there is also another solution to the field equations, which we briefly like to discuss. If $\chi = 0$ there is a solution with $m(r) = 0$. After a rescaling of $q(r)$ one finds exactly the same equations as were obtained in the Nielsen Olesen (NO) model by [7] for the minimal flux $n = 1$, provided we set the value of $\lambda = 2\xi$. Numerical solutions to these equations have been studied before. For a special value of λ one obtains the solutions by solving, Bogomol'nyi type, first order equations, signaling the possibility of extending the model to a super-symmetric one whose super-symmetry increases for this value of the parameter. The residual symmetry of this solution in our model is $\mathbb{Z}_2$. One may wonder whether in our model this is a stable solution. In the case of $\gamma = 0$, i.e. $\chi = 0$, the potential (2) has the form:

$$V = -\mu^2 X + \lambda X^2 \tag{14}$$

The minimum of this potential is obviously given by $X = \frac{\mu^2}{2\lambda}$, with $X = \frac{1}{2}Tr(\Phi)^2$. Written in the components m and q this gives:

$$c_1 m^2 + c_2 q^2 = 1 \tag{15}$$

with $c_1, c_2 > 0$. A simple rescaling of m and q yields: $m^2 + q^2 = 1$. As we require finite energy, this is one of the boundary conditions for the fields m and q at spatial infinity. We now see that the boundary condition of the new solution, $m = 0$, may be continuously changed to one where $q = 0$. But then the higgs field does no longer stabilize the flux, which means that the flux will decay by spreading out and losing more and more energy. Through this process we end up in a "rotated" Alice phase of the theory where the $U(1)$

generator points in the internal direction of the flux we started of with. When going from the $m = 0$ to the $q = 0$ boundary condition we pass an Alice phase whose $U(1)$ generator is perpendicular to the internal direction of the flux we started with. The upshot of these observations is that, if one wants to have a stable Alice flux in an Alice phase, one needs to have $\gamma \neq 0$. Or, stated the other way around, if we do have $\gamma = 0$ and are in an Alice phase with an (unstable) Alice flux, the Alice flux will decay and change the Alice phase into another Alice phase, whose $U(1)$ generator is in the internal direction of the flux we started of with. This concludes what we have to say about conventional Alice electrodynamics, in the remaining sections of the paper we will focus on some alternative Alice models.

2 Alternative Alice models.

In this section we introduce two alternative models, which exhibit an Alice electrodynamic phase. In these alternative models we choose the higgs field(s) in the adjoint (3) representation of $SO(3)$. This obviously means that the higgs field is not single valued in the presence of an Alice flux, but this can be "solved" in two more or less similar ways. One way is to put the internal space of the higgs field (X) on a $\mathbb{Z}_2$ orbifold, i.e. X and $-X$ are identified with each other. The other way is to use (at least) two higgs fields and put the total internal space of these two higgs fields (X and Y) on a S_2 orbifold, i.e. you identify the points (X, Y) and (Y, X).
The action we use for both models is given by:

$$\begin{aligned} S \;=\; & \int d^4x \; \{Tr\; \{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}D^\mu X D_\mu X \; - \frac{1}{2}D^\mu Y D_\mu Y \; + \frac{\hat{\gamma}}{2}[X,Y]^2\} \\ & + \frac{\hat{\lambda}}{4}\{Tr\{X^2 + Y^2\} - f^2\}^2\} \end{aligned} \tag{16}$$

Both theories allow the presence of an Alice flux. In the S_2 model it means that one studies the twisted sector of the theory.

2.1 Alice flux solutions

We now turn to the construction of regular cylindrically symmetric (numerical) solutions corresponding to an Alice flux. At spatial infinity one has $D_\theta X = 0$, implying that the higgs field should have the following form at spatial infinity.

$$X(\theta) = S(\theta)X(0)S^{-1}(\theta) \tag{17}$$

with $S(\theta)$ the same as in (6). The flux associated with $S(\theta)$, is topologically stable if the element of the gauge group associated with $S(2\pi)$ is an element of the disconnected part of the residual gauge group. A simple choice [1] is: $A_\theta = \frac{1}{2er}T_1$. This puts the Alice flux in the internal T_1 direction. Writing the higgs field as $X = x^a T_a$, it follows that $X(\theta)$ has

[1] At this point we can not yet say that this is an element of the disconnected part of the residual gauge group, but this will be done consistently below.

the following form:

$$X(\theta) = e^{\frac{T_1\theta}{2}} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$$

Thus you see that $X(2\pi)$ is given by:

$$X(2\pi) = \begin{pmatrix} x_1(0) \\ -x_2(0) \\ -x_3(0) \end{pmatrix}$$

The same holds, of course, for the other higgs field. Because the two models differ slightly in constructing the ansatz, we will treat them separately for the moment .

The $\mathbb{Z}_2$ model:
For the $\mathbb{Z}_2$ model the boundary condition specified above implies that either $x_1 = 0$ or $x_2 = x_3 = 0$. Only in the first case, however, is $S(2\pi)$ an element of the disconnected part of the gauge group. Thus we have to put $x_1 = 0$. Later we will see that this choice is important in order to obtain first order equations. At this point it is convenient to introduce a different basis for the generators of the gauge group, a basis naturally linked to the orientation of the higgs field. Its elements are given by:

$$S_a(\theta) = e^{\frac{T_1\theta}{2}} T_a e^{\frac{-T_1\theta}{2}} \tag{18}$$

Now we write the higgs field as $X = x^a S_a$, where also in this language one has to put $x_1 = 0$ to secure the possibility of a topological stable solution.
In this model a single higgs field would suffice, but for reasons of similarity we will use two. Our ansatz than reads:

$$\begin{aligned} A_\theta &= \frac{\alpha(r)}{2er} S_1 & (19) \\ X &= a(r) S_3 & (20) \\ Y &= c(r) S_1 & (21) \end{aligned}$$

The S_2 model:
The 'double valuedness' is only allowed if one uses an orbifold interpretation. So we impose a strict relation between X and Y.

$$X(\theta + 2\pi) = \begin{pmatrix} x_1(\theta) \\ -x_2(\theta) \\ -x_3(\theta) \end{pmatrix} = \begin{pmatrix} y_1(\theta) \\ y_2(\theta) \\ y_3(\theta) \end{pmatrix} = Y(\theta)$$

Leading to: $x_1 = y_1$, $x_2 = -y_2$ and $x_3 = -y_3$. Again we are going to work with the twisted generators (18). A consistent ansatz is the following one:

$$\begin{aligned} A_\theta &= \frac{\alpha(r)}{2er} S_1 & (22) \\ X &= a(r) S_3 + c(r) S_1 & (23) \\ Y &= -a(r) S_3 + c(r) S_1 & (24) \end{aligned}$$

For both cases one may insert the appropriate ansatz in the field equations. This yields after a suitable rescaling, the same set of differential equations for both models:

$$\partial_r^2\alpha(r) - \frac{1}{r}\partial_r\alpha(r) = (\alpha(r)-1)a^2(r) \tag{25}$$

$$\begin{aligned}\partial_r^2 a(r) + \frac{1}{r}\partial_r a(r) &= \frac{(\alpha(r)-1)^2}{4r^2}a(r) + \lambda a(r)(a^2(r)+c^2(r)-1)\\ &\quad +\gamma c^2(r)a(r)\end{aligned} \tag{26}$$

$$\partial_r^2 c(r) + \frac{1}{r}\partial_r c(r) = \gamma a^2(r)c(r) + \lambda c(r)(a^2(r)+c^2(r)-1) \tag{27}$$

The asymptotic values of the fields are are as follows:

field	$r\rightarrow 0$	r$\rightarrow\infty$
$\alpha(r)$	0	1
$a(r)$	0	1
$c(r)$	*constant*	0

The boundary conditions are such that $S(2\pi)$ is an element of the disconnected part of the residual gauge group.
We have constructed numerical solutions to these equations, for different values of λ (and γ), with the use of a "shooting" method, see figures 2 and 3. As a matter of fact, we only found solutions for which $c(r)=0$, although our starting values were chosen quite general. This implies that there is no dependence of the solutions we found, on γ.

In fact if $c(r)=0$, the equations become the same as in the case of a NO flux with the critical value of the Landau coupling parameter leading to first order Bogomol'nyi equations. However, there is an important difference with the NO case. The "winding" number of the Alice flux is fractional and equals $n=\frac{1}{2}$, a value which is not admissible in the NO model. This is clearly a consequence of the different breaking schemes of the theories in question.
There is a special role in these theories for the parameter γ, if we set $\gamma=0$ the equations are very similar to the equations (10)-(12) with $\chi=0$. Though γ appears to play no role as long as $c(r)=0$, this is not quite the case. We don't want γ to vanish because than we run more or less into the same problem as in the conventional model for Alice electrodynamics with $\chi=0$. The solution with $c(r)=0$ would still be a solution of the field equations, but the flux would no longer be stable. It would be allowed to decay into the vacuum. In fact in the alternative models it is quite clear what happens at $\gamma=0$, the potential term proportional to γ – assuming it is nonzero – ensures that there is no continuous path in the vacuum manifold connecting the $c=0$ to the $a=0$ boundary condition[2]. If $\gamma=0$ such a path does exist.

There is a simple relation between the $\mathbb{Z}_2$ model with one higgs field and the S_2 model with two higgs fields, in the presence of an Alice string. In the presence of an Alice string the field component of the higgs field parallel to the Alice flux is zero in the $\mathbb{Z}_2$ model, whereas in the S_2 model this is in general only true far away from the core. So, in some sense the $\mathbb{Z}_2$ model is a long wavelength approximation of the S_2 model, but remarkably

[2]Remember, there is also the boundary condition $a^2+c^2=1$.

enough, it does support solutions which are regular everywhere nevertheless. The action of both models becomes the same, up to a rescaling, if the components parallel to the Alice flux, of the higgs fields in the S_2 model, are set equal to zero.

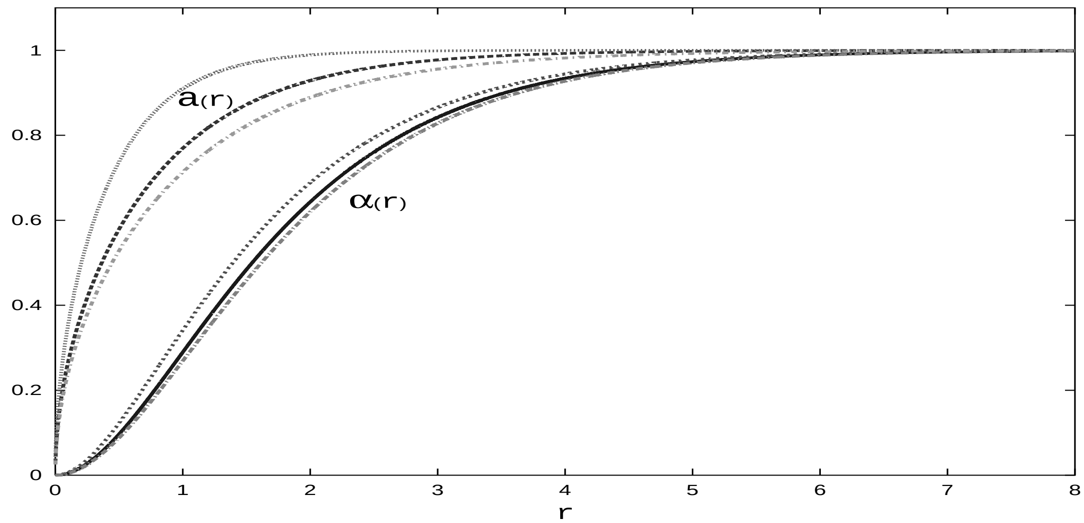

Figure 2: The fields $\alpha(r)$ and $a(r)$ for $\lambda = 2.0...0.5....0.3$.

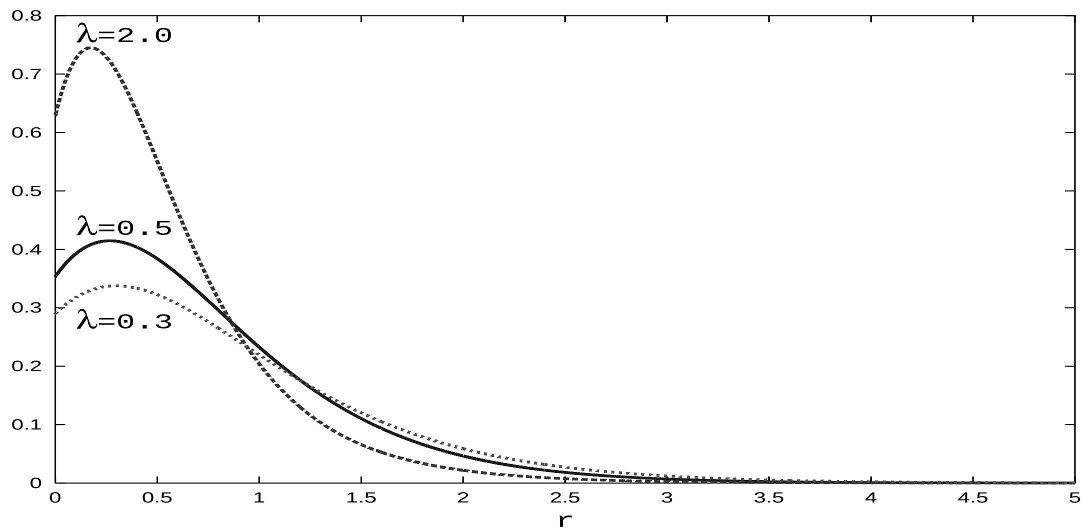

Figure 3: The energy density times r of the Alice flux for $\lambda = 2.0...0.5....0.3$.

2.2 First order equations

As mentioned before, if one sets $c(r) = 0$, the set of equations, (25-27), reduces to the same set that one would obtain in the NO model for a solution with winding number $n = \frac{1}{2}$. It thus appears that one can, in the sector that contains a topologically stable Alice flux, project both theories on a sector of the NO model. This raises the question whether it would be possible to find first order equations in both models. In the $\mathbb{Z}_2$ model, with only a single higgs field, this projection is the clearest. For the rest of this section we will therefore concentrate on this case.

One of the features of the NO theory is that for a certain value of the coupling constant λ, the solutions can be obtained from first order equations. These first order equations can be found *à la* Bogomol'nyi, by rewriting the energy density as a sum of squares plus a topological term. In the case of static solutions, the energy density and the Lagrangian differ only by a sign, implying that extrema of the energy are also extrema of the Lagrangian. Consequently, solutions of minimal energy are stable static solutions of the full set of second order field equations. The energy of the $\mathbb{Z}_2$ model is given by:

$$E = \frac{1}{2}\int d^3x\{\ Tr(\ E_i^2 + B_i^2 + (D_iX)^2 + (D_tX)^2\ ) + \frac{\lambda}{2}(\ (x^1)^2 + (x^2)^2 + (x_3)^2 - 1)\ \} \quad (28)$$

In the static case with no electric fields, in the gauge $A_t = 0$, one has $E_i = \partial_t A_i = 0$ and $D_t X = \partial_t X = 0$, reducing the expression to,

$$E = \frac{1}{2}\int d^3x\{\ Tr(\ B_i^2 + (D_iX)^2\ ) + \frac{\lambda}{2}((x^1)^2 + (x^2)^2 + (x_3)^2 - 1)\ \} \qquad (29)$$

Restricting ourselves to the plane, the energy written in components is given by:

$$\begin{aligned} E &= \frac{1}{2}\int d^2x\{(B_z^1)^2 + (B_z^2)^2 + (B_z^3)^2 + ((D_\nu X)^1)^2 + ((D_\nu X)^2)^2 + ((D_\nu X)^3)^2 \\ &\quad + \frac{\lambda}{2}((x^1)^2 + (x^2)^2 + (x^3)^2 - 1)\}, \end{aligned} \qquad (30)$$

where now the upper label refers to the internal directions (in the normal, non-twisted basis) and the lower label to the spatial directions. From this expression we are unable to obtain first order equations, however, if we restrict ourselves to the subspace of solutions containing an Alice flux, there is something we can do. Let's call the internal direction in which the Alice flux 'points' the '1' direction, so if one is looking for topological stable fluxes one needs to have $x_1 = 0$, as argued before. In that case we may write the energy as,

$$\begin{aligned} E &= \frac{1}{2}\int d^2x\{(B_z^1)^2 + (B_z^2)^2 + (B_z^3)^2 + ([A_1, X]^1)^2 + ([A_2, X]^1)^2 + (\partial_1 x^2 + A_1^1 x^3)^2 \\ &\quad + (\partial_1 x^3 - A_1^1 x^2)^2 + (\partial_2 x^2 + A_2^1 x^3)^2 + (\partial_2 x^3 + A_2^1 x^2)^2 \\ &\quad + \frac{\lambda}{2}((x^2)^2 + (x^3)^2 - 1)\} \end{aligned} \qquad (31)$$

For the case of $\lambda = \frac{1}{2}$ this can be brought into the form:

$$\begin{aligned} E &= \frac{1}{2}\int d^2x\{(B_z^2)^2 + (B_z^3)^2 + ([A_1, X]^1)^2 + ([A_2, X]^1)^2 \\ &\quad + (B_z^1 \pm \frac{1}{2}((x^2)^2 + (x^3)^2 - 1))^2 + ((\partial_1 x^2 + A_1^1 x^3) \mp (\partial_2 x^3 - A_2^1 x^2))^2 \\ &\quad + ((\partial_2 x^2 + A_2^1 x^3) \pm (\partial_1 x^3 - A_1^1 x^2))^2 \pm [A_1, A_2]^1((x^2)^2 + (x^3)^2 - 1)\} \\ &\quad \pm \frac{1}{2}\int d^2x (B_z + [A_1, A_2])^1 \end{aligned} \qquad (32)$$

Still, there appears to be a problem, because there are two terms in this expression of the energy density, which are not squares and as we will show later, only one of them is proportional to the winding number. The other term, $[A_1, A_2]^1((x^2)^2 + (x^3)^2 - 1)$, therefore appears to be problematic. This problem can fortunately be cured rather straightforwardly. Remember that we are already in a 'gauge' $A_t = 0$. where furthermore the static fields are time independent. In this situation the residual gauge freedom of time independent gauge transformations may be used to put the term $[A_1, A_2]^1$ equal to zero. In this gauge the energy density consists only of squares and a term proportional to the winding number. The minimum of the energy is now easily obtained by putting all squares in the energy density equal to zero. This then yields a set of first order equations of which the solutions are also solutions of the full field equations. The first order equations, including the gauge conditions, are:

$$[A_1, A_2]^1 = 0 \tag{33}$$

$$B_z^2 = 0 \tag{34}$$

$$B_z^3 = 0 \tag{35}$$

$$[A_1, X]^1 = 0 \tag{36}$$

$$[A_2, X]^1 = 0 \tag{37}$$

$$(B_z^1 \pm \frac{1}{2}((x^2)^2 + (x^3)^2 - 1) = 0 \tag{38}$$

$$(\partial_1 x^2 + A_1^1 x^3) \mp (\partial_2 x^3 - A_2^1 x^2) = 0 \tag{39}$$

$$(\partial_2 x^2 + A_2^1 x^3) \pm (\partial_1 x^3 - A_1^1 x^2) = 0 \tag{40}$$

The last three equations are identical to those that were obtained in the NO model. The energy of solutions to this set of equations are fully determined by the term $\int d^2x(B_z + [A_1, A_2])^1$ which is proportional to the winding number, as we show next.

The general expression for X in the presence of an Alice flux in the first isospin direction "along" T_1, becomes:

$$X = x(r)e^{2\pi i\chi(\theta)T_1}T_2 = x(r)\cos(2\pi\chi(\theta))T_2 + x(r)\sin(2\pi\chi(\theta))T_3 \equiv aT_2 + bT_3 \tag{41}$$

With $\chi(\theta + 2\pi) = \chi(\theta) + \frac{1}{2}$. For $r \to \infty$ one has $x(r \to \infty) = 1$, and the winding number can be extracted from the asymptotics by:

$$\frac{-i}{2\pi} \oint^{r\to\infty} d\ln(a + i * b) = n = \frac{1}{2} \tag{42}$$

For $r \to \infty$ one also has the spatial covariant derivatives $DX = 0$ or:

$$\partial X = [A, X] \tag{43}$$

or in components:

$$\begin{aligned} [A, X]^1 &= 0 \\ \partial a &= -A^1 b \\ \partial b &= A^1 a \end{aligned}$$

From this one finds $\partial \ln(a + i * b) = i * A^1$, which means that:

$$\frac{-i}{2\pi} \oint^{r\to\infty} d\ln(a + i * b) = \frac{1}{2\pi} \oint^{r\to\infty} d\vec{l}\,\hat{l} \cdot A^1 = \frac{1}{2\pi} \int d^2x(B_z^1 + [A_1, A_2]^1) \tag{44}$$

Thus the rescaled energy of the solutions is equal to $\frac{\pi}{2}$. Note that the above expressions do not look gauge invariant because we are evaluating a gauge invariant expression in a particular gauge.
One should, of course, check whether the first order equations actually do have any solutions of the type we are interested in. By inserting the ansatz used before and putting $c(r)$ to zero, one arrives at the following set of coupled non-linear first order equations.

$$r\partial_r a(r) = \frac{1}{2}(1-\alpha(r))a(r) \tag{45}$$

$$\frac{1}{r}\partial_r \alpha(r) = 1 - a^2(r) \tag{46}$$

Now, these turn out to be a special case of the equations encountered before by De Vega and Schaposnik [7] in their study of the NO model. They where obviously only interested in the case of integer winding number, whereas we are interested in the case of fractional winding number $n = \frac{1}{2}$. The corresponding numerical solution is given in figures 2 and 3.

We have attained our goal of obtaining a set of first order equations, of which the solutions are also static minima of the energy (with no electric fields). As is well-known , first order equations play a deep role in gauge theories. Bogomol'nyi [8] explained, for the NO model, that solutions which come from the first order equations are also minima of the energy, which implied the neutral stability of such solutions. Later it was shown [9] that the occurrence so-called Bogomol'nyi equations is tightly connected to the existence of a super-symmetric extension of the theory. The explicit super symmetry extension of the NO model was given by [10] and in agreement with [9] showed that the first order equations indeed follow from an increase of super-symmetry. In our models we also found first order equations whose solutions are also solutions to the full set of second order field equations. We showed that the solutions are also minima of the energy. This obviously raises the question if these first order solutions can also be explained by an increase of super symmetry of a super-symmetric extension of our models. A superficial analysis suggests that this is not the case, basically because we can only recover the Bogomol'nyi argument within the context of a very restrictive ansatz. In this respect the situation is similar to that encountered in the study of regular $\mathbb{Z}_N$ monopoles [11].

3 Conclusion

In this paper we proposed two new models which both possess an Alice electrodynamics phase. For both models we constructed solutions corresponding to a topologically stable Alice flux. We found a way to project the theories on the Nielsen Olesen model and, in that way, obtained first order equations. Solutions to these first order equations corresponding to minima of the energy (without electric fields) were constructed numerically.
We close with a brief remark concerning the zero modes of our solution. E.Weinberg [12] showed that in the NO model, for the critical value $\lambda = \frac{1}{2}$, a flux with winding number n has $2n$ zero modes. These modes are interpreted as being the positions of the unit fluxes. At first sight this appears to give problems for the case of $n = \frac{1}{2}$, but carefully redoing section IV of the article mentioned, in particular using the fact that our fields are allowed to be double valued, one may show that the answer for $n = \frac{1}{2}$ is that there are again two

zero modes, as one would expect.

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