Alice Electrodynamics

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November 1997

Ah! ne jamais sortir des Nombres et des Êtres!
— Baudelaire

Abstract

This thesis studies Alice electrodynamics. Alice electrodynamics is similar to ordinary electrodynamics except that it features topological defects like magnetic monopoles and so-called “Alice strings”. The interaction between Alice strings and electric and magnetic charges is of a topological nature and highly unusual. In this thesis we make a detailed study of these unusual properties, in particular the occurence of non-localized “Cheshire charge”. We construct exact (though numerical) solutions of the pure Alice string and of a new electrically charged Alice string, which challenges the standard interpretation of “Cheshire charge”.

Contents

Chapter 1

Introduction

1.1 Topological defects

Topological defects are closely related to the phenomenon of spontaneous symmetry breaking. They arise as stable, non-perturbative, collective excitations in the medium and they carry a “charge” that is conserved for topological reasons. Defects can occur as localized particle-like, string-like or planar-like objects, or — when localized in both space and time — they can be understood as quantum mechanical tunneling processes.

Topological defects have been extensively studied both in condensed matter physics and in particle physics. Examples of defects in condensed matter systems are: the domain walls of a ferromagnet; the magnetic flux lines in a type II superconductor; and the quantized vortex lines in superfluid ^{4}He. The defects can be produced in systems in thermal equilibrium or in systems out of equilibrium, i.e., during phase transitions (through “pair creation”).

Research on topological defects in particle physics started in the early seventies when defect solutions were discovered in spontaneously broken Yang-Mills theories, notably the Nielsen-Olesen vortex-line in 1973 and the ’t Hooft-Polyakov monopole in 1974. Since then, much has become known about the properties of defects, their physical interactions with ordinary particles and with other defects, and the formation and evolution of defects.

It turns out that all Grand Unified Theories of elementary particles necessarily contain defect solutions, in particular magnetic monopoles. If GUTs correctly describe nature, these defects must somehow exist. But topological defects of GUTs are very heavy and so it is unlikely that they can be made by accelerators or supernovas. However, a key notion of grand unification is that the known symmetries of elementary particles resulted from a larger symmetry group G after a series of spontaneous symmetry breakings,

$$G \to H \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)_{\rm em}. \tag{1.1.1}$$

In a cosmological context, this implies that the early universe has gone through a number of phase transitions. During such a cosmological phase transition, topological defects could have been formed. Furthermore, it is possible that the evolution and structure of the universe has been significantly influenced by these defects. For example, it has been suggested that cosmic strings may have acted as seeds for the formation of galaxies and other large scale structures in the present day universe.

Therefore, the study of topological defects is essential to gain more insight into GUTs and the early history of the universe. Apart from this, topological defect are interesting objects in their own right. They are very different from more familiar physical objects and they give rise to a rich variety of unusual physical phenomena.

1.2 Alice electrodynamics

In this thesis we focus on a particular type of defect, the so-called Alice strings. Alice strings are solutions to a minimal nonabelian extension of electrodynamics. Locally, Alice electrodynamics appears to be indistinguishable from standard electrodynamics, but Alice electrodynamics exhibits global and topological properties that are highly nontrivial.

For instance, if a charged particle is transported around an Alice string, the sign of the charge flips. Similarly, a magnetic monopole that goes around an Alice string, is turned into an anti-monopole. These interactions between charges and an Alice string are of a topological nature as they only depend on the number of times the charge goes around the string and are independent of the distance between the charge and the string.

Closed loops of Alice string can have charged excitations, and a charged particle that goes through an Alice loop transfers charge to the loop. The charged excitations of an Alice loop can be peculiar: the loop can have a long-range electric field from which its charge can be inferred, but the source of the charge cannot be localized, neither on the string nor in its vicinity. Such non-localizable charge has been called "Cheshire charge".

In this thesis we focus on the simplest model which exhibits Alice electrodynamics. It is not a "real world" model, but it should be noted that Alice strings could also occur in Grand Unified Theories. For example, it has been shown that some $SO(10)$ cosmic strings are Alice strings that can carry non-localizable $SU(3)_{color}$ Cheshire charge.

In this thesis we propose a new interpretation of charged Alice strings. Briefly put, we think that the electric-magnetic dual symmetry is a priori not broken by the Alice string, but may be broken by either an electric or magnetic charge excitation of the string core. These excitations may be obtained by imposing suitable boundary conditions. We can paraphrase this view in the following three questions: 1) Can Alice strings have excitations where the electric/magnetic charge is localized in the string core? 2) Can Cheshire charge be understood as electric/magnetic flux confined by a magnetic/electric supercurrent? 3) Are the electric and magnetic excitations each other's dual image? In the last chapters we try to verify this view.

1.3 Outline of the thesis

In chapter two we set the stage and briefly summarize the essential ingredients of gauge field theories, spontaneous symmetry breaking and topological defects. Two classic defect solutions are discussed in more detail: the Nielsen-Olesen vortex-lines and the 't Hooft-Polyakov magnetic monopole.

In chapter three we introduce the minimal Alice model and analyse the basic properties of the model's defect solutions: Alice strings and magnetic monopoles.

In chapter four we construct an explicit solution for an Alice string. We derive the exact field equations for this solution, determine its asymptotics in great detail and present numerical solutions.

In chapter five we describe how electric charge can be transferred to an Alice string and we discuss the elusive concept of Cheshire charge.

In chapter six we argue that an Alice string can carry electric charge localized at its core. We construct an ansatz for a charged Alice string, derive the exact field equations, determine the asymptotic behaviour of a charged string and present numerical solutions. We show that the gauge invariant characteristics of the solution are consistent. The results in this chapter are new and provide a sensible alternative to the esoteric option of Cheshire charge.

In chapter seven we give a topological definition of the magnetic charge carried by an Alice loop. We show that if a magnetic monopole is carried through a loop, magnetic charge is transferred to the loop. Finally, we construct an ansatz for a string with magnetic charge and magnetic current.

1.4 Units and notation

The units employed in this thesis are the fundamental units ($\hbar = c = 1$). Space-time indices are denoted by Greek letters and run through the four values 0, 1, 2, 3. Spatial indices are denoted by Latin letters from the middle of the alphabet with values 1, 2, 3. Implicit summation over repeated indices is assumed, unless stated otherwise. The metric signature is taken to be $(+,-,-,-)$.

Chapter 2

Topological defects in gauge field theories

In this chapter we present a concise review of the main features of topological defects in gauge field theories. We enumerate the essentials of gauge field theories and spontaneous symmetry breaking; we discuss the main ideas of topological defects, touching lightly on homotopy theory; and finally, we tackle two simple defect solutions: the Nielsen-Olesen vortex and the 't Hooft-Polyakov magnetic monopole.

This review provides the background material necessary to render this thesis more or less self-contained. It is summarily and tailored to suit our needs. This chapter should therefore not be taken for an introduction to gauge field theory and topological defects. To this end, many excellent textbooks and review articles exist.

The literature on gauge field theory is large and ever-growing. The present writer happened to benefit much from Ryder (1985), a clear introductory text.

On topological defects in particular, see: Coleman (1985), one of the first review articles, a pleasant reading and very rewarding; Preskill (1987), short but wide-ranging, and especially of interest since it contains an elementary account of the $\mathbb{Z}_2$ vortex and of superconducting strings, two topics which we will treat in later chapters; Vilenkin and Shellard (1994), mainly on (cosmic) strings, its convenient bibliography runs to twenty-five pages; Rajaraman (1982), focuses on the quantisation of topological defects, but the first three chapters on classical defects contain much information; van der Meer (1997), on magnetic monopoles, has an eye for little details that are often unheeded: glancing through its pages can save one from tiring hours of tracking down a faulty sign.

Most of these texts also treat the basics of homotopy theory. A more detailed exposition of homotopy theory can be found in Mermin (1979) and, for those who are not daunted by abstract mathematics, Nash and Sen (1983).

2.1 Local gauge field theories

We consider field theories which are invariant under an N-dimensional Lie group G of local gauge transformations, and which contain an n-component scalar field ϕ_i. We assume that G is simple, compact and connected. The elements of the group can then be written as

$$g = \exp(-i\omega_a L^a), \tag{2.1.1}$$

where the ω_a are arbitrary real numbers and the L^a are the N group generators. The generators L^a generate the Lie algebra of G and satisfy the commutation relations

$$\left[L^a, L^b\right] = -if_{abc}L^c \tag{2.1.2}$$

where the constants f_{abc} are the structure constants of G. An n-dimensional matrix representation of G is a mapping from G onto a group of matrix operators acting on an n-dimensional vector space, such that group multiplication is preserved. We can generate a matrix representation by picking N hermitian matrices T^a that satisfy the commutation relations (2.1.2). The N-dimensional representation generated by the structure constants themselves, $(T^a)_{bc} = -if_{abc}$, is known as the adjoint representation.

A function $g(x)$ from coordinate space to G, defines a local gauge transformation. The scalar fields transform through the action of an n-dimensional matrix representation of G,

$$\phi_i(x) \to \phi_i'(x) = S_{ij}(g(x))\phi_j(x). \tag{2.1.3}$$

The transformation matrix S can be written as

$$S(x) = \exp(-i\theta_a(x)T^a), \tag{2.1.4}$$

where T^a are $n \times n$ matrix representations of the group generators, and $\theta_i(x)$ are functions of space time. For notational convenience, we identify the group G with the representation $S(g)$. Thus the abstract objects $g(x)$ and L^a are interpreted as $S(x)$ and T^a, respectively.

To construct a lagrangian which is invariant under local gauge transformations we introduce N vector fields A^a_μ, one for each group generator,

$$A_\mu = A^a_\mu T^a. \tag{2.1.5}$$

These are the gauge fields. They transform according to

$$A_\mu(x) \to S(x)A_\mu(x)S(x) + ie^{-1}S(x)^{-1}\partial_\mu S(x), \tag{2.1.6}$$

where e is the gauge coupling constant. Further, we define the field strength $F_{\mu\nu} = F^a_{\mu\nu} T_a$ as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\, f_{abc} A^b_\mu A^c_\nu, \tag{2.1.7}$$

and the gauge-covariant derivative

$$D_\mu \boldsymbol{\phi} = (\partial_\mu - ie A^a_\mu T^a)\boldsymbol{\phi}. \tag{2.1.8}$$

These objects transform as

$$D_\mu \boldsymbol{\phi} \to S D_\mu \boldsymbol{\phi}, \tag{2.1.9}$$
$$F_{\mu\nu} \to S F_{\mu\nu} S^{-1}. \tag{2.1.10}$$

Now we can construct the gauge-invariant lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} D_\mu \boldsymbol{\phi} D^\mu \boldsymbol{\phi} - V(\boldsymbol{\phi}), \tag{2.1.11}$$

where the potential function $V(\boldsymbol{\phi})$ must be invariant under gauge transformations, i.e. $V(S\boldsymbol{\phi}) = V(\boldsymbol{\phi})$. The Euler-Lagrange equations of motion are

$$D^\mu D_\mu \boldsymbol{\phi}^a = -\frac{\partial V}{\partial \boldsymbol{\phi}^a}, \tag{2.1.12}$$
$$D^\mu F^a_{\mu\nu} = e\varepsilon^a{}_{bc} (D_\nu \boldsymbol{\phi}^b) \boldsymbol{\phi}^c. \tag{2.1.13}$$

The hamiltonian is obtained by a Legendre transformation of the lagrangian,

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\phi}}} \boldsymbol{\phi} - \mathcal{L}, \tag{2.1.14}$$

yielding the generic energy of a non-abelian field theory

$$\begin{aligned} E &= \int \mathcal{H}\, dx^3 \\ &= \int \left(\frac{1}{2} (\vec{E^a} \cdot \vec{E^a} + \vec{B^a} \cdot \vec{B^a}) + \frac{1}{2} D_\mu \boldsymbol{\phi} D_\mu \boldsymbol{\phi} + V(\boldsymbol{\phi}) + W \right) dx^3 \end{aligned} \tag{2.1.15}$$

with

$$W = -F^a_{i0} (\partial_i A^a_0 + e\varepsilon_{abc} A^b_i A^c_0) + e[A_0, \boldsymbol{\phi}] D_0 \boldsymbol{\phi}.$$

The ground state of the theory is a field configuration for which, up to a gauge transformation, the scalar field $\boldsymbol{\phi}$ is a constant and a minimum of V and the gauge fields A_μ are zero. Since we can always add a constant to V, we will assume that V is normalized such that it is always greater than or equal to zero. Then the energy of the ground states is zero.

2.2 Spontaneous symmetry breaking

The gauge symmetry G is said to be spontaneously broken if the ground state of the theory is not invariant under all gauge transformations. For such a ground state $\boldsymbol{\phi}_0 \neq 0$, we can define a subgroup H of G which consists of all the elements of G that leave $\boldsymbol{\phi}_0$ unchanged,

$$H = \{g \in G \,|\, S(g)\boldsymbol{\phi}_0 = \boldsymbol{\phi}_0\}\,. \tag{2.2.1}$$

The group H is called the unbroken subgroup of G with respect to $\boldsymbol{\phi}_0$, or the residual symmetry group, or the stabilizer of $\boldsymbol{\phi_0}$). The generators t^α of H all annihilate $\boldsymbol{\phi}_0$

$$t^\alpha \boldsymbol{\phi}_0 = 0. \tag{2.2.2}$$

We can choose the generators T^a such that the t^α are a subset of the T^a. The t^αs are referred to as the unbroken generators of G, while the remainder of the T^as are referred to as the broken generators.

Since V is invariant under G, every state $S(g)\boldsymbol{\phi}_0$ with $g \in G$ is also a ground state of the theory. We will assume that *all* ground states are of the form $S\boldsymbol{\phi}_0$ (i.e. we exclude accidental degeneracy and non-gauge internal symmetries). As a consequence, we can identify the set of ground states with the coset space[1] G/H.

The physical fields are the oscillations about the ground state. Representing $\boldsymbol{\phi}$ as $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \boldsymbol{\phi}'$, we obtain the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\phi}')^2 - \frac{1}{2}\mu_{ij}^2 \phi_i' \phi_j' - \frac{1}{4} M_{ab}^2 A_\mu^a A^{b\mu} + \mathcal{L}_{int}, \tag{2.2.3}$$

where

$$\mu_{ij}^2 = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0}, \tag{2.2.4}$$

is the scalar field mass matrix,

$$M_{ab}^2 = e^2 (T^a T^b)_{ij} \phi_{0i} \phi_{0j}, \tag{2.2.5}$$

is the vector field mass matrix, and $\mathcal{L}_{int}$ includes the cubic and higher order terms in $\boldsymbol{\phi}'$ and A_μ.

Since $\boldsymbol{\phi}_0$ is a minimum of V,

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} = 0 \tag{2.2.6}$$

[1] Also known as the quotient space. Here, it is the set of all equivalence classes $[g] = \{gh \,|\, h \in H\}$ with $g \in G$.

and the eigenvalues of the mass matrix μ_{ij}^2 are non-negative. The potential is invariant under gauge transformations, so

$$\frac{\partial V}{\partial \phi_i} T_{ij}^a \phi_j = 0. \tag{2.2.7}$$

Differentiating this and using (2.2.6), we obtain

$$\mu_{ij}^2 T_{jk}^a \phi_{0k} = 0. \tag{2.2.8}$$

All vectors $T^a\boldsymbol{\phi}$ formed from the broken generators ($T^a\boldsymbol{\phi} \neq 0$) are linearly independent, so u_{ij}^2 must have a zero eigenvalue for each broken generator. If the gauge symmetry were *global*, these zero eigenvalues would have corresponded to massless Goldstone bosons. In a local gauge theory however, the Goldstone bosons disappear. The components of the vector ϕ_i' in the subspace defined by the vectors $T^a\boldsymbol{\phi}_0$, are unphysical; this becomes manifest in the unitary gauge, the gauge in which these components vanish. The remaining scalar fields are, in general, massive.

Inspecting (2.2.5), we see that the gauge fields associated with the broken generators have acquired non-zero masses, while the gauge fields of the unbroken group H remain massless. The massive gauge fields have, so to speak, absorbed the would-be Goldstone bosons. This is the famous Higgs mechanism. In this context, the scalar fields are often called Higgs fields.

2.3 Topological defects and homotopy theory

Some field theories possess striking classical solutions: solutions that are non-dissipative and have a finite, localized energy. These solutions are called solitary waves or — shortly and loosely — solitons.[1]

Here we will only be concerned with time-independent, topological soliton solutions of gauge field theories. Topological solitons are non-dissipative due to the topology of the phase space of the system.[2] If the phase space is made up of connected components separated by infinitely high potential barriers, then a field configuration that starts in a given component can never reach another component without an infinite cost of energy. Hence, corresponding with the component of phase space in which they lie, topological solitons can be assigned a conserved index or charge. This charge conservation does not arise out of a symmetry of the theory — as for example ordinary charge conservation does — but it is a consequence of the connectedness of the phase space. Hence the adjective "topological": the topological charge of a soliton is conserved owing to a topological conservation law.

[1] A detailed discussion of the meaning of these two terms can be found in Rajaraman (1982).

[2] By phase space, we mean the space consisting of all pairs $(\pi(\boldsymbol{x}), \phi(\boldsymbol{x}))$ for which the energy is finite.

The connectedness of the phase space of a gauge theory is directly related to the topology of its ground state space G/H. To understand this, consider the energy functional (2.1.15). In a gauge where the time components of the gauge potentials, A_0^a, vanish, the energy of a field configuration is the sum of five non-negative terms

$$E = \int \left[\frac{1}{2}(\vec{E^a} \cdot \vec{E^a} + \vec{B^a} \cdot \vec{B^a}) + \frac{1}{2}\partial_0 \boldsymbol{\phi} \partial_0 \boldsymbol{\phi} + \frac{1}{2} D_i \boldsymbol{\phi} D_i \boldsymbol{\phi} + V(\boldsymbol{\phi}) \right]. \qquad (2.3.1)$$

Each of these terms must be finite if the total energy is to be finite. In particular, the Higgs field must approach a zero of the potential at spatial infinity. For example, in (2+1) dimensions we must have

$$\lim_{r \to \infty} \boldsymbol{\phi}(t, r, \theta) = \boldsymbol{\phi}_\infty(t, \theta), \qquad (2.3.2)$$

with

$$V(\boldsymbol{\phi}_\infty) = 0. \qquad (2.3.3)$$

The function $\boldsymbol{\phi}_\infty(t, \theta)$ need not be constant, but may depend non-trivially on θ provided that $D_\theta \boldsymbol{\phi}$ vanishes sufficiently fast at infinity. Thus, in (2+1) dimensions every field configuration, at any time t, is associated with a mapping of a circle into the ground state space G/H

$$\boldsymbol{\phi}_\infty : S^1 \to G/H. \qquad (2.3.4)$$

Since time evolution is continuous, two field configurations lie in the same connected component of phase space if they can be continuously deformed into each other without an infinite cost of energy. Two field configurations that have the same mapping ϕ_∞ associated with them can be continuously deformed into one another, since a deformation in the finite part of space only takes a finite amount of energy. On the other hand, two field configurations that have different mappings associated with them, can be continuously deformed into one another if and only if their mappings can be continuously deformed into one another. From this it follows that to determine whether two field configurations lie in the same component of phase space, we need only determine whether their associated mappings can be continuously deformed into each other.

A similar argument applies to (1+1) and (3+1) dimensions. So, in general, we can classify the components of phase space and thus the possible soliton solutions by classifying the distinct mappings

$$\boldsymbol{\phi}_\infty : S^n \to G/H. \qquad (2.3.5)$$

(The symbol S^n denotes the n-dimensional sphere.)

A branch of mathematics is concerned with just this kind of topological classification: homotopy theory. We briefly consider a few of its terms and theorems pertinent to our concerns.

All non-singular mappings from S_n into a space $\mathcal{M}$ can be classified into homotopy classes. Two mappings are in the same class if they can be continuously deformed into one another. The classes are endowed with a group structure by appropriately defining the composition of two mappings. For two closed paths in $\mathcal{M}$ (i.e. two mappings from S_1 into $\mathcal{M}$), their composition may be defined as the path that traverses the two paths in succession.

The group that classifies mappings from S_1 into $\mathcal{M}$ is called the fundamental group of the manifold $\mathcal{M}$; it is denoted by $\pi_1(\mathcal{M})$. The second homotopy group $\pi_2(\mathcal{M})$ is the set of homotopically equivalent classes of maps from S^2 into the manifold $\mathcal{M}$.[1]

Topological soliton solutions are solutions to the field equations that cannot be continuously deformed to the ground state. The homotopy groups $\pi_n(G/H)$ classify the possible soliton solutions.[2] A soliton with a non-trivial mapping of S^1 into G/H is called a vortex in two spatial dimensions and a string in three spatial dimensions. A soliton which maps S^2 non-trivially into G/H is called a monopole. These soliton solutions can be seen as topological defects in the Higgs field. In their core, the scalar field deviates from the ground state and thus respects symmetries different from those respected by the ground state.

The determination of the first and second homotopy group can often be simplified by employing the first fundamental theorem

$$\pi_1(G/H) \cong \pi_0(H), \tag{2.3.6}$$

or the second fundamental theorem

$$\pi_2(G/H) \cong \pi_1(H_0), \tag{2.3.7}$$

where H_0 is the component of the unbroken group H that is connected to the identity. These reductions are only valid if G is connected and simply connected (i.e. $\pi_1(G) \cong I$). It is however possible to embed any compact Lie group G in a larger group $\tilde{G}$, the universal covering group of G, which is simply-connected. If we have enlarged G to $\tilde{G}$, the unbroken group H must also be appropriately enlarged to $\tilde{H}$ by including any additional elements in $\tilde{G}$ that leave the ground state ϕ_0 invariant. These additions factor out of the ground state space G/H.

Elegant and revealing a topological analysis may be, it is always incomplete. It provides little information about the detailed structure of a particular soliton.

[1] In fact, these two homotopy groups are defined with reference to a base point x_0 in $\mathcal{M}$. But if $\mathcal{M}$ is connected, homotopy groups based at different points are identical, and we can omit the base point.

[2] However, in general there is no one-to-one correspondence between the elements of the homotopy group and the types of solitons. This is only so, for $\pi_1(G/H)$ if $\pi_1(G/H)$ is abelian; and for $\pi_2(G/H)$ if $\pi_1(G/H) \cong I$. See for example **?**) or Vilenkin and Shellard (1994).

Furthermore, topological analysis may establish the existence of non-dissipative solutions, but it cannot prove that these solutions are time-independent and stable. A field configuration must of course remain in the topological sector in which it starts, but it does not follow that for every sector there will be a *time-independent* solution; nor, that if a time-independent solution exists, it will be *stable*.

In the next two sections we will briefly consider two classical examples of topological defects: the Nielsen-Olesen vortex and the 't Hooft-Polyakov monopole. These defect solutions may serve as a prelude to the more complicated vortex and monopole solutions of the Alice model discussed in the following chapters.

2.4 Nielsen-Olesen vortex lines

A simple example of a vortex occurs in the abelian-Higgs model: the Nielsen-Olesen (1973) vortex lines. The model has a complex scalar field ϕ and a $U(1)$ gauge symmetry. The lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{D_\mu\phi}D^\mu\phi - V(\phi), . \tag{2.4.1}$$

We will suppose that the potential has the form

$$V(\phi) = \frac{1}{4}\lambda(\overline{\phi}\phi - \eta^2)^2, \tag{2.4.2}$$

with positive constants λ and η. The minima of the potential make up a circle $\|\phi\| = \eta$, and the ground state of the theory can be expressed as $\phi = \eta e^{i\vartheta}$ with ϑ an arbitrary phase. Thus, the $U(1)$ symmetry is spontaneously broken and the Higgs mechanism has occurred. The vector boson has acquired a mass $m_v = e\eta$, there is no Goldstone boson and the remaining scalar field has a mass $m_s = \sqrt{\lambda}\eta$.

The model contains vortex solutions. To see this, consider the properties of finite-energy field configurations. The energy of a field configuration is comprised of three nonnegative terms

$$E = \int \left[\frac{1}{2}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}) + \frac{1}{2}D_i\boldsymbol{\phi}D_i\boldsymbol{\phi} + V(\boldsymbol{\phi})\right]. \tag{2.4.3}$$

In order that the total energy is finite, each of these terms must be finite. The third term is finite if the potential approaches zero at spatial infinity. Therefore any finite energy solution must have

$$\phi_\infty(\theta) = \eta e^{i\vartheta(\theta)}. \tag{2.4.4}$$

where the phase angle ϑ may depend on θ since finiteness of the third term places no restriction on the phase of ϕ_∞. Thus, every finite-energy field configuration is associated with a mapping from the circle at spatial infinity to the circle of minima. In going along the circle at spatial infinity, the field ϕ can wrap a number of times around the circle of minima, i.e. the phase ϑ of ϕ can develop a winding

$$n = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\vartheta}{d\theta} d\theta, \tag{2.4.5}$$

where the winding number n is an integer. Since the winding number is an integer, field configurations with different winding numbers cannot be continuously deformed into one another. Thus, the abelian-Higgs model can possess vortex solutions and these vortex solutions are classified by their non-zero winding number (the vacuum has zero winding number). This result is succinctly expressed in the formula

$$\pi_1(G/H) \cong \pi_1(S_1) = \mathbb{Z}. \tag{2.4.6}$$

Since the covariant derivative, $D_\theta\phi = (\partial_\theta - ieA_\theta)\phi$, must also vanish at infinity, the gauge fields must be

$$A_\theta = \frac{1}{er}\frac{d\vartheta}{d\theta}. \tag{2.4.7}$$

The gauge field has a "pure gauge"-form. This permits the field $F_{\mu\nu}$ to decay sufficiently fast at large r for the first term in (2.4.3) to be finite. If $n \neq 0$, the gauge field cannot be pure gauge everywhere. Using Stokes' theorem

$$\Phi_B = \int \boldsymbol{B}\cdot \boldsymbol{dS} = \oint \boldsymbol{A}\cdot \boldsymbol{dl} = \frac{2\pi n}{e} \tag{2.4.8}$$

we find from the asymptotics of the gauge field (2.4.7), that a vortex has a quantised magnetic flux through the plane, Φ_B.

Nielsen and Olesen studied the lagrangian (2.4.1) within the context of high-energy physics, but it is interesting to note that this lagrangian is in fact the relativistic generalization of the Landau-Ginsburg theory of type-II superconductors in a magnetic field. Thus, (2.4.8) expresses the well known fact that magnetic flux can only penetrate a superconductor in quantised (Abrikosov) flux lines.

Let us now turn to the construction of explicit solutions for the vortex lines. To this end, we can try a cylindrically-symmetric ansatz of the form

$$\phi(r,\theta) = f(r)\mathrm{e}^{in\theta}, \tag{2.4.9}$$

$$A_\theta(r,\theta) = n\frac{\alpha(r)}{er}, \tag{2.4.10}$$

$$A_r = A_0 = 0. \tag{2.4.11}$$

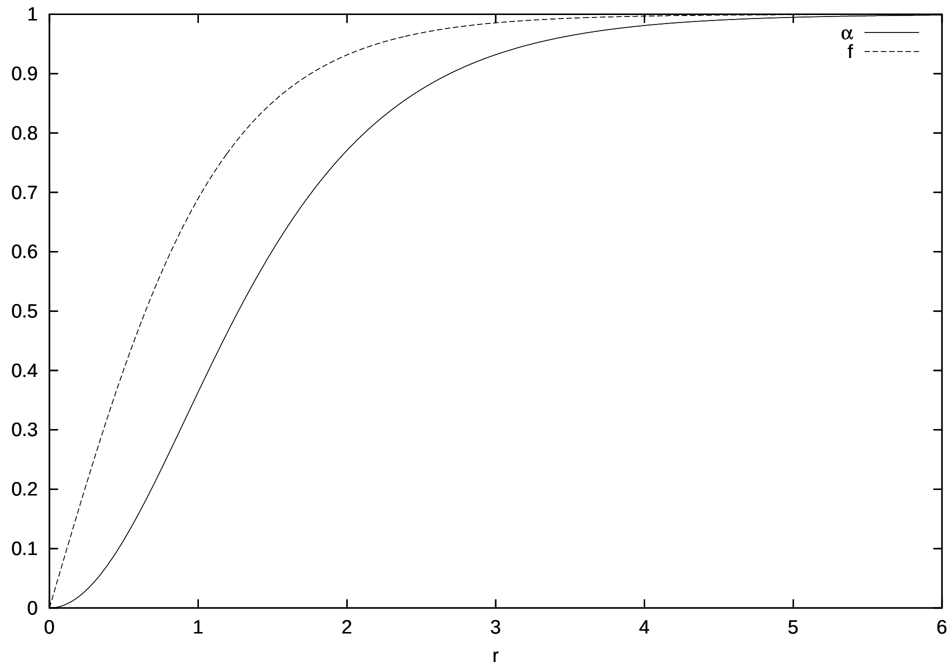


Figure 2.1: Functions $\alpha(r)$ and $f(r)$ for the Nielsen-Olesen vortex solution with $n = 1$ and $\beta = 1$.

The boundary conditions (2.4.4) and (2.4.7) imply

$$f(r) \to \eta, \qquad \text{and} \qquad \alpha(r) \to 1, \tag{2.4.12}$$

as $r \to \infty$. To ensure regularity at the origin,

$$f(r) \to 0, \qquad \text{and} \qquad \alpha(r) \to 0, \tag{2.4.13}$$

as $r \to 0$. Substituting the ansatz in the Euler-Lagrange equations yields

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} = 2f^2(\alpha - 1), \tag{2.4.14a}$$

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = \frac{n^2 f}{r^2}(\alpha - 1)^2 - \beta f(f^2 - 1), \tag{2.4.14b}$$

where we have rescaled to dimensionless variables

$$\phi \to \eta^{-1}\phi, \qquad A^\mu \to \eta^{-1}A^\mu, \qquad r \to e\eta r, \tag{2.4.15}$$

leaving the ratio of the Higgs and vector masses,

$$\beta \equiv m_s^2/m_v^2 = \lambda/2e^2 \tag{2.4.16}$$

as the only significant parameter in the model.

No analytic solution is known to these equations for any β, but it can be shown that solutions, if they exist, will approach the boundary values (2.4.12) exponentially fast (Perivolaropoulos, 1993),

$$f(r) \to 1 + c_f r^{-1/2} \mathrm{e}^{-\sqrt{2\beta} r} \qquad \beta < 4, \tag{2.4.17a}$$

$$f(r) \to 1 - \frac{c_\alpha^2 \mathrm{e}^{-2\sqrt{2}r}}{2(\beta - 4)r} \qquad \beta > 4, \tag{2.4.17b}$$

and

$$\alpha(r) \to 1 + c_\alpha r^{1/2} \mathrm{e}^{-\sqrt{2}r}. \tag{2.4.17c}$$

This shows that the energy of a vortex is confined within a finite-sized core.

Numerical work has supported the existence of vortex solutions with these asymptotic properties, where at least the $n = 1$ solution seems to be stable. Figure 2.1 shows the numerically calculated solution for $n = 1$ and $\beta = 1$.

In three spatial dimensions, the vortex solution may be thought of as a cross section of an infinite string. Equation (2.4.3) is then interpreted as the energy per unit length of the string.

2.5 The 't Hooft-Polyakov monopole

A three dimensional topological defect solution was first discovered by 't Hooft (1974) and, independently, by Polyakov (1974). This solution is characterized by a long-range magnetic monopole field, which prompted 't Hooft to call it a magnetic monopole.

't Hooft and Polyakov considered the Georgi-Glashow model, which possesses gauge symmetry $G = SU(2)$ with a Higgs field $\boldsymbol{\phi}$ in a triplet representation. The $SU(2)$ symmetry is spontaneously broken to $U(1)$ due to the potential

$$V(\boldsymbol{\phi}) = \frac{1}{4}\lambda(\phi^a\phi^a - \eta^2)^2, \tag{2.5.1}$$

which is zero for $\phi^a\phi^a = \eta^2$. Hence, the ground state space is

$$G/H = SU(2)/U(1) \cong S^2. \tag{2.5.2}$$

Since the group $SU(2)$ is simply-connected, we may use the second fundamental theorem (2.3.7) to find

$$\pi_2(SU(2)/U(1)) \cong \pi_1(U(1)) \cong \mathbb{Z}. \tag{2.5.3}$$

The monopole solutions of the model can classified by a topological index $q \in \mathbb{Z}$.

The 't Hooft-Polyakov monopole has $q = 1$. In a particular gauge, this field configuration will at spatial infinity take the form

$$\phi^a_\infty = \eta \frac{x^a}{r}. \tag{2.5.4}$$

The direction of $\phi^a(\boldsymbol{x})$ in internal space is aligned with the spatial direction $\hat{\boldsymbol{x}} = \boldsymbol{x}/x$, so the Higgs field points "radially outwards", which prompted Polyakov to call the configuration a hedgehog.

In order that the field configuration has finite energy, the covariant derivative $D_i\boldsymbol{\phi}$ must approach zero at large r sufficiently fast. Therefore, the gauge fields are

$$\begin{aligned} A^a_{i,\infty} &= -\varepsilon_{aij}\frac{x^j}{er^2}, \\ A^a_{0,\infty} &= 0. \end{aligned} \tag{2.5.5}$$

The interpretation of the gauge fields requires special attention. The unbroken group $U(1)$ can be regarded as the gauge group of electromagnetism. For example, consider the vacuum configuration. For this configuration, there is a gauge such that the Higgs field points in the same direction everywhere, say $\boldsymbol{\phi}(\boldsymbol{x}) = (0, 0, a)$. The massless particle corresponding to the field A^3_μ must now be identified with the photon, and the field $F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ with the electromagnetic field.

For the monopole configuration there is no gauge where the Higgs field points in the same direction everywhere. We therefore need an expression of the electromagnetic field that is $SU(2)$-gauge invariant. 't Hooft suggested

$$\mathcal{F}_{\mu\nu} = \frac{\phi^a}{\|\boldsymbol{\phi}\|}F^a_{\mu\nu} + \frac{1}{e\|\boldsymbol{\phi}\|^3}\varepsilon_{abc}\phi^a(D_\mu\phi^b)(D_\nu\phi^c). \tag{2.5.6}$$

This expression reduces to the standard expression in the gauge where $\boldsymbol{\phi}$ points in the same direction everywhere. It can be verified that $\mathcal{F}_{\mu\nu}$ satisfies the ordinary Maxwell equations at all points where $\boldsymbol{\phi} \neq 0$.

Now, substituting (2.5.5) in (2.5.6), we find the monopole to have a long range magnetic field,

$$\boldsymbol{B} = \frac{g\boldsymbol{r}}{r^3}, \tag{2.5.7}$$

with a magnetic charge,

$$g = \frac{1}{e}. \tag{2.5.8}$$

We can conclude that the field configuration with asymptotic form (2.5.4–2.5.5) carries quantized magnetic charge. This magnetic charge appears as a topological charge, and elements of the homotopy group $\pi_2(G/H)$ can be identified with the particular values the magnetic charge can take. Thus, the topological conservation law expressed by (2.5.3) is equivalent to the conservation of magnetic charge.

We can obtain an ansatz for a static solution by multiplying (2.5.4) and (2.5.5) with functions of r,

$$\phi^a_\infty = \eta h(r)\frac{x^a}{r}, \tag{2.5.9}$$

$$A^a_{i,\infty} = -\left[1 - K(r)\right]\varepsilon_{aij}\frac{x^j}{er^2}, \qquad A^a_{0,\infty} = 0. \tag{2.5.10}$$

The functions $h(r)$ and $K(r)$ are subject to the boundary conditions

$$h(r) \to 1, \qquad K(r) \to 0, \tag{2.5.11}$$

as $r \to \infty$. Substituting this ansatz in the field equations for this model, we can obtain regular solutions. We will not pursue this matter here.

With these two examples we have come to the end of this preliminary survey of topological defects in gauge theories. In the next chapter we introduce Alice strings, which exhibit a more intricate topological structure.

Chapter 3

An introduction to Alice electrodynamics

There exist gauge field theories where charge conjugation is a *local* symmetry. In these theories, the unbroken gauge group H contains both a electromagnetic $U(1)$ group generated by a charge operator Q, and an element X of a disconnected component of H, such that $XQX^{-1} = -Q$. The discrete symmetry X "flips" the electromagnetic subgroup $U(1)$. As a consequence, the absolute sign of the charges and the electromagnetic fields is undetermined.

In addition, such theories contain topological string-like solutions which have the puzzling property that if a charged particle is taken around the string, its charge changes sign. These strings were first studied by Schwarz (1982). Schwarz dubbed them "Alice strings", as they appear to act as a charge-conjugating looking-glass (Carroll, 1871).

Models that possess Alice strings, also possess magnetic monopoles. A magnetic monopole that goes once around an Alice string is turned into an anti-monopole.

In this chapter we introduce the simplest model that contains an Alice string: a $SO(3)$ gauge theory with the Higgs field belonging to the 5-dimensional representation of $SO(3)$. In the first section we discuss the model, the symmetry breaking and the electrodynamics it gives rise to. In the second section we consider the Alice string and in the final section we look at the magnetic monopoles of the model.

A considerable number of articles have been devoted to Alice strings. Particularly useful articles include: Bucher *et al.* (1992), Preskill and Krauss (1990) and Alford *et al.* (1991).

The type of Alice string we study in this thesis is not expected to exist in nature since it is known that charge conjugation is not an exact symmetry. Still, Alice strings could conceivably play a role in Grand Unified Theories. For example, there might be an exact discrete symmetry in nature that interchanges ordinary particles with "mirror particles". In addition, Bucher and Goldhaber

(1994) have shown that, without the need to introduce new discrete symmetries, some $SO(10)$ cosmic strings are non-abelian Alice strings carrying non-localizable $SU(3)_{color}$ Cheshire charge.

3.1 The model

The model we consider has a gauge group $SO(3)$, like the Georgi-Glashow model, but its Higgs field transforms as the 5-dimensional irreducible representation of $SO(3)$. We describe the Higgs field as a real symmetric traceless 3×3 matrix, Φ^{ab}, which transforms as

$$\Phi \to S \Phi S^{-1}, \tag{3.1.1}$$

where the S's are the rotation matrices in the adjoint representation of $SO(3)$.

We can construct a potential function $V(\Phi)$ such that the zeros of V take the form

$$\Phi^{ab} = \beta\,(3\hat{\phi}^a\hat{\phi}^b - \delta^{ab}), \tag{3.1.2}$$

with $\hat{\boldsymbol{\phi}}$ a unit vector. The ground states break the $SO(3)$ symmetry. To determine the unbroken symmetry group, let us consider the ground state for which $\hat{\boldsymbol{\phi}}$ lies along the z-axis,

$$\Phi_0 = \beta\,\mathrm{diag}(1, 1, -2). \tag{3.1.3}$$

The Higgs field Φ_0 leaves unbroken a subgroup of $SO(3)$ which is locally isomorphic to $SO(2)$; it consists of rotations about the z-axis, generated by

$$Q = T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.1.4}$$

and can be parametrized as

$$H_c = \{\exp(\theta T_3)\,|\,0 \leq \theta < 2\pi\}\,. \tag{3.1.5}$$

But in addition, the unbroken group contains a disconnected component generated by

$$X = \mathrm{diag}(1, -1, -1). \tag{3.1.6}$$

This component consists of rotations by 180° about axes that lie in the xy-plane,

$$H_d = \{X\exp(\theta T_3)\,|\,0 \leq \theta < 2\pi\}\,. \tag{3.1.7}$$

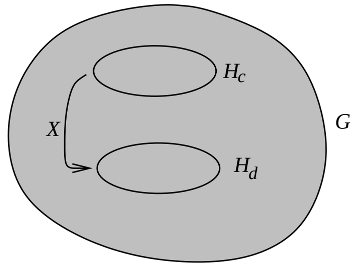


Figure 3.1: The two components H_c and H_d embedded in G.

So the full unbroken group H is isomorphic to $O(2)$. Figure 3.1 represents the group space schematically. We can express the symmetry breaking as

$$SO(3) \to U(1) \rtimes \mathbb{Z}_2. \tag{3.1.8}$$

where $\rtimes$ denotes a semi-direct product, since the $U(1)$ generator T_3 does not commute with the rotation X.

The unbroken $U(1)$ symmetry may be identified with the gauge group of electromagnetism. If the system is in the ground state (3.1.3), then the electromagnetic fields are given by the T_3-component of the field tensor,

$$F_{\mu\nu} = F^a_{\mu\nu} T^a, \tag{3.1.9}$$

But since the unbroken group in addition contains a *local* $\mathbb{Z}_2$ symmetry, the model globally deviates from ordinary electrodynamics. The X operator anti-commutes with the charge operator Q,

$$XQX^{-1} = -Q, \tag{3.1.10}$$

implying that a gauge transformation generated by X will flip the sign of the electromagnetic fields. The X operator acts, so to speak, as a charge-conjugation operator.

To illustrate this, consider the following example. Suppose there is a linear electrical field, as sketched in figure 3.2a. If we perform a local transformation X in a region R, the field lines in this region will turn into the opposite direction (figure 3.2b). At first sight, this might seem to lead to an inconsistency: how can the electric field ever act on a charge if its direction can be reversed at will? But of course, the matter fields also transform under X. Suppose we add charged particles to the theory, described by fields ψ^a. The fields are locally transformed by

$$\boldsymbol{\psi}(x) \to X\boldsymbol{\psi}(x) \qquad (x \in R). \tag{3.1.11}$$

Since the charge operator anti-commutes with X (3.1.10),

$$\mathrm{e}^{iQ} X\boldsymbol{\psi}(x) = X\mathrm{e}^{-iQ}\boldsymbol{\psi}(x) \qquad (x \in R), \tag{3.1.12}$$

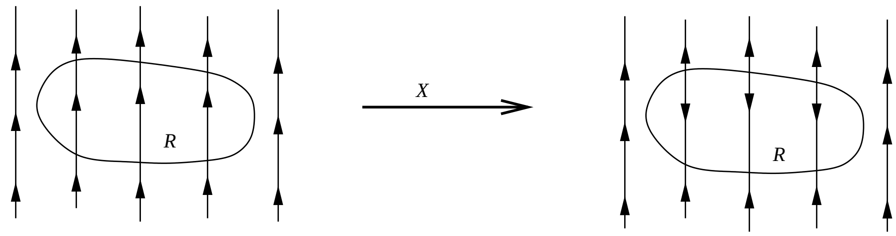


Figure 3.2: The local gauge transformation X in region R flips the sign of the electric field lines in R.

we see that a particle entering the region R will have the sign of its charge flipped. So the sign change of the electric fields and the sign change of the charges cancel out. The second description then is consistent, and completely equivalent to the first one — naturally, as the second description is obtained from the first one by a gauge transformation.

Thus, in this model, the sign of the charges and of the electromagnetic fields is gauge-dependent. We can flip the sign of any charge by making a local gauge transformation. But although the absolute sign of charges is undetermined, the relative sign is still well-defined: it can simply be inferred from the long-range electromagnetic force between the charges, because this force is gauge-invariant. But as we will see, even this observation has to be qualified if Alice strings are present.

3.2 Alice strings

We now turn to the topological classification of the possible defect solutions in this model. We need to establish whether there are loops or spheres in the ground state manifold that cannot be contracted to a point. The ground states in this model can be represented by three dimensional vectors of unit length: $\hat{\boldsymbol{\phi}}$. The space of these unit vectors is isomorphic to a two dimensional sphere, but since the vectors $\hat{\boldsymbol{\phi}}$ and $-\hat{\boldsymbol{\phi}}$ define the same ground state, as follows from (3.1.2), antipodal points on the sphere are identified. Thus the ground state space G/H is a two-sphere with antipodal points identified. It is isomorphic to $\mathbb{RP}^2$, the real projective space in three dimensions.

Non-contractible loops in G/H correspond to possible vortex solutions (or, in three space dimensions, string solutions). On a two-sphere, any closed path is contractible to a point. On $\mathbb{RP}^2$ however, there are two possibilities: either the path is manifestly closed, or it connects two antipodal points. A manifestly closed path can obviously be contracted to a point, but a path connecting two antipodal points can not. Hence the mappings from S^1 into G/H fall into two homotopy classes,

$$\pi_1(G/H) = \pi_1(SU(2)/U(1) \times_{S.D.} \mathbb{Z}_2) \cong \mathbb{Z}_2. \tag{3.2.1}$$

With the non-trivial element of π_1, there may correspond a string solution. In the next chapter, we will show that indeed such a string solution can be constructed. This string is a $\mathbb{Z}_2$ string. The difference between a $\mathbb{Z}_2$ and a $\mathbb{Z}$ string (such as the Nielsen-Olesen string) becomes manifest when we look at the composition of two closed paths in G/H. The composition of two paths can be defined as the path that traverses the two paths in succession. In the Nielsen-Olesen model, if we compose two paths, one having winding number n and the other having winding number m, the resulting path will have winding number $n+m$. But in the present model, if we compose two non-contractible paths, the resulting path is contractible to a point and consequently we have $1+1=0$, i.e. the fluxes add modulo two. Thus a configuration of two $\mathbb{Z}_2$ strings can be continuously deformed to a ground state configuration. As a consequence, the combined string can decay, since there is no topological constraint that forces the energy to remain localized in the strings.

For a string solution, the field $\boldsymbol{\phi}$ rotates along a circle at spatial infinity and on its return points in a direction opposite to its starting direction. The Higgs field must return to its original value and this condition is fulfilled since $\hat{\boldsymbol{\phi}}$ and $-\hat{\boldsymbol{\phi}}$ represent the same Higgs field. A possible string configuration is shown in figure 3.3, where the orientation of the Higgs field is represented by bidirectional arrows.

The rotation of the Higgs field can be described by a transformation $U(\theta)$,

$$\Phi_\infty(\theta) = U(\theta)\Phi_\infty(0)U(\theta)^{-1}. \tag{3.2.2}$$

In order for the energy of the string to be finite, the covariant derivative of the Higgs field must be zero at infinity,

$$D_\mu \Phi = 0. \tag{3.2.3}$$

Solving this equation, we find that

$$U(\theta) = P\exp\left(e\int_0^\theta A_\theta r d\theta\right), \tag{3.2.4}$$

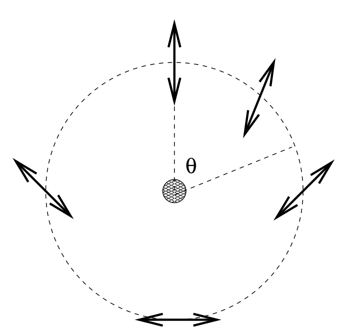


Figure 3.3: An Alice string configuration. The orientation of the Higgs field is represented by the bidirectional arrows.

where P denotes path ordering.

The Higgs field is parallel transported by $U(\theta)$. For $\theta = 2\pi$, the field must return to its original value and therefore $U(2\pi)$ must be an element of $H(\theta = 0)$. If Φ is a string configuration, then $U(2\pi)$ is an element of H_d and $U(\theta)$ is a path in $SO(3)$ from the connected component to the disconnected component of the unbroken group $O(2)$ (see figure 3.1). Thus we find that the string carries a $\mathbb{Z}_2$ "magnetic" flux, taking values in H_d.

A particle that is transported around an Alice string experiences a gauge transformation by $U(2\pi)$. Because $U(2\pi)$, being an element of H_d, anti-commutes with the charge operator Q (3.1.10), the particle returns with its charge reversed. This statement may be elucidated by considering the behaviour of the unbroken group H as it is parallel transported around the string. At each point on the circle at spatial infinity, there is a subgroup $H(\theta)$ embedded in G that consists of the elements of G that leave $\Phi(\theta)$ unchanged (see figure 3.4). These subgroups are all isomorph to $H(0)$ and are related by the rotation U,

$$H(\theta) = U(\theta)H(0)U(\theta)^{-1}. \tag{3.2.5}$$

While it is of course true for the whole group that

$$H(2\pi) = U(2\pi)H(0)U(2\pi)^{-1} = H(0), \tag{3.2.6}$$

the analogous relation does *not* hold for the individual generators of H. In particular, the charge operator rotates as

$$S_3(\theta) = U(\theta)T_3U(\theta)^{-1}. \tag{3.2.7}$$

and hence is double-valued in the background of a string. Specifically, for $\theta = 2\pi$ we obtain

$$S_3(2\pi) = -T_3 \equiv -Q, \tag{3.2.8}$$

and so the generator Q is rotated into $-Q$ upon parallel transport around the string.

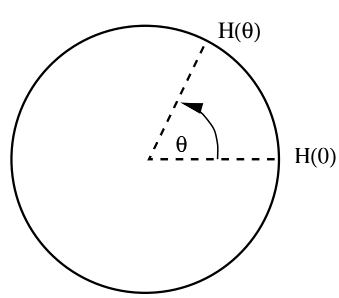


Figure 3.4: At each point of the circle there is an unbroken group $H(\theta)$ that leaves the field $\Phi(\theta)$ at that point unchanged.

The embedding of H in G around the string is analogous to a Möbius strip. The $U(1)$ subgroups $H(\theta)$ can be represented as undirected lines in $\mathbb{R}^3$ that coincide with the axes of the rotation of the $U(1)$ subgroups. Choosing a generator $Q(\theta)$ for $H(\theta)$ at each θ is equivalent to choosing a direction for each of these lines. As θ varies from 0 to 2π, the lines are twisted into a Möbius strip. There is no continuous way to choose a direction on each of them.

Actually, this fact implies that the local $U(1)$ symmetry cannot be globally extended: there are no global $U(1)$ symmetries in the background of a string. It is easy to see this. Global transformations are singled out as those transformations that transform the fields at each point *in the same way*. The qualification "in the same way" is understood as: "being the same after parallel transport". Suppose we start with a local transformation $S(x_0)$ at a particular point x_0. Extending this transformation globally, the transformation $S(x_1)$ at any point x_1 would be obtained by parallel transporting $S(x_0)$ from x_0 to x_1. But this will succeed only if parallel transport is path independent and, as we know, in this model parallel transport is *not* path independent: transport around the string inevitably induces a rotation of Q.

It might be thought that since the sign of a charge is not a gauge invariant notion, the sign flipping of a charge will not yield a physical effect. Remarkably enough, this appears not to be the case. Consider the following *gedanken experiment*. Suppose we have two charges of equal charge q and an Alice string. We can bring the charges together and establish whether they attract or repel each other. Suppose they attract. Then, let us take the particle with charge $+q$ around the string while leaving the other behind. When reunited, the charges will repel!

The gedanken experiment raises an intriguing question: assuming that total charge is conserved, what happened to the $2q$ units of charge we seem to have lost? Some charge appears to be left behind while the particle passed "behind" the string. How can that be? We will return to this puzzle in chapter 5.

The experiment reveals another important fact: in the background of a string, it is meaningless to speak of two charges having the same or opposite sign *globally*; we can only determine the relative sign of two charges locally, when we bring them together and the outcome clearly depends on the path that we choose in uniting them.

3.3 Magnetic monopoles

Magnetic monopoles are associated with non-contractible two-surfaces in G/H. In the present model where $G/H = \mathbb{RP}^2$, magnetic monopoles exist. We can easily classify them by using the second fundamental theorem (2.3.7)

$$\pi_2(G/H) \cong \pi_2(SU(2)/U(1) \rtimes \mathbb{Z}_2) \cong \pi_1(U(1)) \cong \mathbb{Z}. \qquad (3.3.1)$$

Thus a monopole is characterized by an integer winding number n. However, since the unbroken group contains the charge conjugation operator X, monopoles with opposite winding, a monopole and an anti-monopole, are gauge equivalent. It follows that the absolute sign of the magnetic charge of a monopole is undetermined. We can simply use positive integers to classify the monopoles.

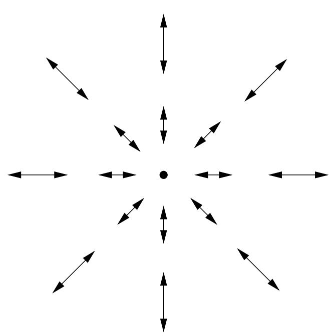
Yet, we can determine the relative sign of two monopoles because of the Coulomb force between them. For example, if we bring two monopoles m and n together, we obtain either the $m + n$ or $|m - n|$ element. If Alice strings are present, the result of this process depends on the path taken by the monopoles relative to the Alice strings. It appears that a monopole which goes once around an Alice string, is turned into an anti-monopole. In chapter 7 we give a topological analysis of this process.

It appears that a magnetic monopole can be deformed to a closed loop of Alice string. To see this, consider figure 3.5a which schematically represents a monopole configuration. As before, the orientation of the Higgs field is represented by bidirectional arrows. Continuity requires that the Higgs field is zero at the core of the monopole.

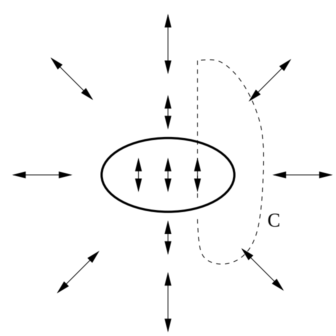
We can "punch a hole" in the monopole core and deform it to a small loop. In the core of the string the Higgs field must still be zero, but the field can assume a non-zero value in the space "surrounded" by the loop (see 3.5b). This deformation is allowed because the Higgs field is represented by a bidirectional arrow. It is not possible to deform a 't Hooft-Polyakov monopole to a loop, since here the Higgs field has a direction and so continuity requires that the field is zero within a sphere-like core.

Following a closed path C that goes around the loop, the Higgs field is rotated by 180° (see 3.5b). This implies that the loop constitutes a closed Alice string. Furthermore, since the long range fields have not been changed, the loop carries magnetic charge as well.

This argument suggests that a closed Alice string can carry magnetic charge.



(a)



(b)

Figure 3.5: A magnetic monopole configuration (a) can be deformed to a magnetically charged loop of Alice string (b). The orientation of the Higgs field is represented by bidirectional arrows.

Of course, it remains to be shown whether such an object really corresponds to a solution of the field equations. We return to this question in chapter 7.

Chapter 4

Explicit Alice string solutions

In the preceding chapter we considered a simple non-abelian gauge theory. We showed, using topological arguments, that the theory contains an Alice string. In this chapter we construct explicit solutions for the simplest string in this model. In itself of interest, this information will also be helpful to gain more insight into the theory's unusual electromagnetic properties.

The procedure followed in this chapter is general and rather straightforward. First, we construct a simple expression for the fields at spatial infinity. This expression must satisfy the energy constraints, in order that the string energy is finite. With the help of this expression and some symmetry considerations, we construct an ansatz for the string solution. Substituting the ansatz in the Euler-Lagrange field equations, we arrive at the field equations for the ansatz. These allow us to determine the asymptotic behaviour of the fields at the origin and at infinity. As we do not know how to solve the equations analytically, we study them numerically and thus obtain numerical solutions which are regular everywhere.

The first two sections of this chapter rely on an article by Shankar (1976) about $SO(3)$ monopoles. In the course of a general survey, Shankar constructs an ansatz for an Alice string[1], though without being aware of the string's peculiar electromagnetic properties. (Note the dates: Shankar's article was published six years before Schwarz's (1982) seminal article on Alice strings.) To our knowledge, so far no one has studied the exact regular solutions for the Alice string.

4.1 The fields at infinity

The simple model we use has, as will be recalled, gauge group $SO(3)$ and a Higgs field Φ that transforms as the 5-dimensional irreducible representation of $SO(3)$. We express the Higgs field as a real symmetric traceless 3×3 matrix, Φ^{ab}. The

[1] As a historical note, we mention that the correct ansatz was communicated to him by F.A. Bais (private communication), who also pointed out the sign ambiguity for the charge.

lagrangian density is:

$$\mathcal{L} = \frac{1}{8}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{4}\,\mathrm{Tr}(D_\mu\Phi D^\mu\Phi) - V(\Phi), \tag{4.1.1}$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - e[A_\mu, A_\nu], \\ A_\mu &= A_\mu^a T^a, \\ D_\mu\Phi &= \partial_\mu\Phi - e[A_\mu^a T^a, \Phi]. \end{aligned}$$

The 3×3 generators $T^a_{ij} = \varepsilon_{aij}$ are normalized such that $\mathrm{Tr}(T^aT^b) = -2\delta_{ab}$. (The term representing the kinetic energy of the Higgs field has factor 1/4 instead of the usual 1/2 because, the field being a symmetric matrix, tracing doubles the contribution of every field component.) Since in this chapter we look for static solutions, the energy is (cf. (2.1.15))

$$E = \int \left(\frac{1}{2}(\vec{E}^a\cdot\vec{E}^a + \vec{B}^a\cdot\vec{B}^a) + \frac{1}{4}\,\mathrm{Tr}(D_i\Phi D_i\Phi) + V(\Phi)\right) dx^3, \tag{4.1.2}$$

with the usual definitions of the electric and magnetic fields: $E_i^a = F_{0i}^a$ and $B_i^a = \varepsilon_{ijk}F_{jk}^a$.

The requirement that the string energy is finite imposes conditions on the asymptotic field configuration. Far from the core the Higgs field Φ must in every direction approach a vacuum value to minimize the potential term. Also, the covariant derivative of the Higgs field must vanish sufficiently fast at infinity. We first consider the potential term.

The most general, renormalizable potential is:

$$V = -\tfrac{1}{2}\mu^2\ \mathrm{Tr}\,\Phi^2 - \tfrac{1}{3}\gamma\ \mathrm{Tr}\,\Phi^3 + \tfrac{1}{4}\lambda\,(\mathrm{Tr}\,\Phi^2)^2, \tag{4.1.3}$$

since $\mathrm{Tr}\,\Phi^4$ and $\det(\Phi)$ can be expressed in these three terms (Georgi and Glashow, 1972). Now suppose the field has a vacuum expectation value Φ_0. By choosing a suitable gauge we can take Φ_0 to be diagonal:

$$\Phi_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix}. \tag{4.1.4}$$

In order that Φ_0 is an extremum, we must have

$$\left.\frac{\partial V(\Phi)}{\partial\Phi}\right|_{\Phi=\Phi_0} = 0. \tag{4.1.5}$$

This condition yields two equations

$$\frac{\partial V}{\partial a} = 0 = [-\mu^2 - \gamma b + 2\lambda(a^2 + ab + b^2)](2a + b), \tag{4.1.6a}$$

$$\frac{\partial V}{\partial b} = 0 = [-\mu^2 - \gamma a + 2\lambda(a^2 + ab + b^2)](2b + a). \tag{4.1.6b}$$

So, either $(2a + b) = 0$, or $(2b + a) = 0$, or $a = b$. Choosing $a = b$ gives:

$$\Phi_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix}, \tag{4.1.7}$$

where

$$a = b = \frac{\gamma \pm \sqrt{\gamma^2 + 24\mu^2\lambda}}{12\lambda}. \tag{4.1.8}$$

It can easily be checked that Φ_0 is a minimum and not a maximum of the potential.

The ground state Φ_0 is invariant under rotations about the T_3-axis, and invariant under rotations by 180° about axes that lie in the T_1T_2-plane. So, the unbroken subgroup is[1] $H = U(1) \rtimes \mathbb{Z}_2$, in accordance with our analysis in section 3.1. The other two options, $(2a + b) = 0$ and $(2b + a) = 0$, single out T_1, respectively T_2 as rotation axis. These are completely equivalent.

It is possible to express the Higgs field as the symmetric product of two isovectors $\vec{\phi}_1$ and $\vec{\phi}_2$

$$\Phi^{ab} = \phi_1^a\phi_2^b + \phi_2^a\phi_1^b - \tfrac{2}{3}\delta^{ab}(\vec{\phi}_1 \cdot \vec{\phi}_2). \tag{4.1.9}$$

Under gauge transformations, the two fields transform as:

$$\vec{\phi}_1 \to S\vec{\phi}_1, \tag{4.1.10a}$$

$$\vec{\phi}_2 \to S\vec{\phi}_2, \tag{4.1.10b}$$

with $S \in SO(3)$. This notation will turn out to be useful for later purposes, but it can also be used to make the aforesaid even more clear.

The transformation equations (4.1.10) plainly show that in general the field Φ has no axis of invariance; if you rotate around $\vec{\phi}_1$ you affect $\vec{\phi}_2$ and vice versa. There is a $O(2)$ symmetry only if $\vec{\phi}_1$ and $\vec{\phi}_2$ are (anti-)parallel. Of course, this is exactly what happens

$$\vec{\phi}_1 = \vec{\phi}_2 = \sqrt{3a}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.1.11}$$

[1]That is, for at least a range of parameters of $V(\Phi)$. For example, equations (4.1.6) imply that γ cannot taken to be zero. See also condition (4.3.14) on page 37.

Rotations about the T_3-axis leave ϕ_1 and ϕ_2 invariant. Rotations by 180° about axes that lie in the T_1T_2-plane change the sign of both ϕ_1 and ϕ_2, but leave the Higgs field invariant since since the Higgs field is the product of the two iso-vectors.

For a field configuration to have finite energy, the fields must at spatial infinity satisfy the equation

$$D_\mu \Phi = 0. \tag{4.1.12}$$

The general solution to this equation is

$$\Phi(x_1) = [P\mathrm{e}^{e\int_{x_0}^{x_1} A_\mu dx^\mu}]\Phi(x_0)[P\mathrm{e}^{-e\int_{x_0}^{x_1} A_\mu dx^\mu}], \tag{4.1.13}$$

where the integral is along an arbitrary path from x_0 to x_1 and path ordered by P. Thus the field at point x_1 is dependent on the field at point x_0.

Since we are interested in string solutions, we impose cylindrical symmetry. Equation (4.1.12) then reduces to

$$D_\theta \Phi = 0, \tag{4.1.14}$$

because one can always choose a gauge in which $A_r = 0$. Thus, at spatial infinity the Higgs field rotates as

$$\Phi(\theta) = S(\theta)\Phi(0)S(\theta)^{-1}, \tag{4.1.15}$$

with

$$S(\theta) = \exp\left[e\int_0^\theta rA_\theta d\theta\right], \tag{4.1.16}$$

which follows from (4.1.13) if we take $\theta_0 = 0$.

The asymptotic field configuration (4.1.15) is associated with a topologically stable string if $S(2\pi)$ is an element of H_d. A simple and convenient choice is: $S(2\pi) = \exp(T_1\theta/2)$. Thus, the asymptotic form of a string configuration is

$$\Phi = \mathrm{e}^{T_1\theta/2}\Phi_0\mathrm{e}^{-T_1\theta/2}, \tag{4.1.17a}$$

$$A_\theta = \frac{1}{2er}T_1. \tag{4.1.17b}$$

This configuration carries a $\mathbb{Z}_2$ magnetic flux

$$\Phi_B = \int B ds = \int A_\theta r d\theta = \frac{\pi}{e}T_1. \tag{4.1.18}$$

4.2 The ansatz

The next step in our investigation is to find an ansatz that satisfies the field equations. As a first guess, one might just take the fields at infinity (4.1.17) and multiply each field with a function depending on r. This yields an ansatz which is practically equivalent to the ansatz for the Nielsen-Olesen vortex (see section 2.4). But if we substitute this ansatz in the field equations, we get stuck with three inconsistent equations. Apparently, the Alice string structure is essentially more complicated than the Nielsen-Olesen vortex structure. (Schwarz's (1982) statement to the contrary is mistaken.)

Considering that the Higgs field is matrix-valued and has, in diagonal form, *two* independent components (4.1.4), the obvious next try would be to multiply these components each with a different function of r. Thus we get the following ansatz,

$$\Phi = \mathrm{e}^{T_1\theta/2} \begin{pmatrix} m'(r) & 0 & 0 \\ 0 & q'(r) & 0 \\ 0 & 0 & -m'(r)-q'(r) \end{pmatrix} \mathrm{e}^{-T_1\theta/2}, \tag{4.2.1a}$$

$$A_\theta = \frac{\alpha(r)}{2er} T_1. \tag{4.2.1b}$$

This ansatz does indeed yield three consistent differential equations. Further analysis shows it to be the desired ansatz.

This trial and error method to produce the ansatz might not satisfy the reader. The method seems arbitrary and gives us hardly an understanding of why the ansatz must have the form it has. Therefore we will now sketch a more general constructive procedure to generate the appropriate ansatz. The procedure and its application to the present case are described in respectively Bais (1980) and Shankar (1976).

We are looking then for the simplest ansatz with the proper topology. To find this ansatz, we impose a set of symmetries that will strongly simplify the form of the ansatz while making it still possible to get the proper topology. Since the solution should be static and cylindrically symmetric, we may impose invariance under

1. t- and z-translations;
2. rotations around the z-axis, $R(\theta) = \exp[\theta(J_z - \frac{1}{2}T_1)]$;
3. parity, $P = P_s \exp(\pi T_3)$, where P_s operates in coordinate space ($\theta \to -\theta, z \to -z$).

It can easily be checked that the asymptotic form of the fields (4.1.17) has, at least, these symmetries.

Imposing symmetry (1) leaves us with Φ^{ab}, A_r^a, A_θ^b, $(a, b = 1, 2, 3)$, which may be functions of r and θ. If we take $\theta = 2\pi$ in $R(\theta) = \exp[\theta(J_z - \frac{1}{2}T_1)]$, we see that $x_1 \to x_1$, $x_2 \to -x_2$ and $x_3 \to -x_3$ in isospace. This implies that Φ^{12}, Φ^{13}, A_θ^2, A_θ^3 are zero. Invariance under $R(\theta)$ for all θ demands that A_r^1 and A_θ^1 are functions only of r. Parity forces A_r^1 to be zero. So we can parameterize

$$A_\theta^1 = (1/2er)\alpha(r). \tag{4.2.2}$$

Now consider the remaining Φ^{11}, Φ^{23}, Φ^{22} and Φ^{33}. We change the cartesian indices $\hat{x}_2$ and $\hat{x}_3$ for those of $\hat{x}_\pm = (\hat{x}_2 \pm i\hat{x}_3)/\sqrt{2}$. The latter transform simpler under $R(\theta)$:

$$\mathrm{e}^{-T_1\theta/2}\, \hat{x}_\pm\, \mathrm{e}^{T_1\theta/2} = \pm i \mathrm{e}^{\mp i\theta/2}\, \hat{x}_\pm.$$

Invariance under $R(\theta)$ then requires

$$\begin{aligned}
\Phi^{11}(r,\theta) &= m(r),\\
\Phi^{++}(r,\theta) &= \frac{3}{2}\mathrm{e}^{-i\theta}q(r),\\
\Phi^{--}(r,\theta) &= \frac{3}{2}\mathrm{e}^{i\theta}q^*(r).
\end{aligned}$$

The factor $\frac{3}{2}$ is there for convenience. The trace condition gives us $2\Phi^{+-} + \Phi^{11} = 0$. Under P, $\Phi^{++}(r,\theta) \to \Phi^{--}(r,-\theta) = \frac{3}{2}q^*(r)\mathrm{e}^{-i\theta}$; so $q = q^*$. Returning to cartesian components we have

$$\Phi = \begin{pmatrix} m & 0 & 0 \\ 0 & -\frac{1}{2}m + \frac{3}{2}q\cos\theta & \frac{3}{2}q\sin\theta \\ 0 & \frac{3}{2}q\sin\theta & -\frac{1}{2}m - \frac{3}{2}q\cos\theta \end{pmatrix}.$$

Finally then, the complete ansatz is[1]

$$\Phi = m(r)\Phi_1 + q(r)\mathrm{e}^{T_1\theta/2}\Phi_2\mathrm{e}^{-T_1\theta/2}, \tag{4.2.3a}$$

$$A_\theta = \frac{\alpha(r)}{2er}T_1, \tag{4.2.3b}$$

where

$$\Phi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}. \tag{4.2.3c}$$

[1] The parameterization differs slightly from (4.2.1). The present parameterization is more convenient. It turns out that Φ has a non-zero part at the origin; this non-zero part is completely captured by $m(r)$.

The field equations corresponding to lagrangian (4.1.1) are:

$$\text{for } a = 1, 2 \qquad D^\mu D_\mu \Phi^{aa} - D^\mu D_\mu \Phi^{33} = -2\left(\frac{\partial V}{\partial \Phi^{aa}} - \frac{\partial V}{\partial \Phi^{33}}\right), \tag{4.2.4a}$$

$$\text{for } a \neq b \qquad D^\mu D_\mu \Phi^{ab} = -2\frac{\partial V}{\partial \Phi^{ab}}, \tag{4.2.4b}$$

$$D^\mu F_{\mu\nu} = -e((D_\nu \Phi) T^c \Phi) T^c. \tag{4.2.4c}$$

Substituting the ansatz, we get the following system of coupled nonlinear ordinary differential equations

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} = 9e^2 q^2 (\alpha - 1), \tag{4.2.5a}$$

$$\frac{d^2 q}{dr^2} + \frac{1}{r}\frac{dq}{dr} = \frac{(\alpha-1)^2 q}{r^2} - 2\mu^2 q + 2mq\gamma + \lambda(9q^2 + 3m^2)q, \tag{4.2.5b}$$

$$\frac{d^2 m}{dr^2} + \frac{1}{r}\frac{dm}{dr} = -2\mu^2 m + \gamma(3q^2 - m^2) + \lambda(9q^2 + 3m^2)m. \tag{4.2.5c}$$

It is convenient to rescale the fields and the r-coordinate,

$$q \to \frac{\sqrt{\lambda}}{\mu} q, \qquad m \to \frac{\sqrt{\lambda}}{\mu} m, \qquad A_\mu \to \frac{\sqrt{\lambda}}{\mu} A_\mu, \qquad r \to \frac{e\mu}{\sqrt{\lambda}} r, \tag{4.2.6}$$

thereby making these quantities dimensionless. We define two new parameters

$$\xi = \frac{\lambda}{e^2}, \qquad \chi = \frac{\gamma\sqrt{\lambda}}{\mu e^2}. \tag{4.2.7}$$

Finally, the set of dimensionless equations is

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} = 9q^2(\alpha - 1), \tag{4.2.8a}$$

$$\frac{d^2 q}{dr^2} + \frac{1}{r}\frac{dq}{dr} = \frac{(\alpha-1)^2 q}{r^2} + \xi(9q^2 + 3m^2 - 2)q + 2\chi m q, \tag{4.2.8b}$$

$$\frac{d^2 m}{dr^2} + \frac{1}{r}\frac{dm}{dr} = \xi(9q^2 + 3m^2 - 2)m + \chi(3q^2 - m^2). \tag{4.2.8c}$$

These equations agree with those Shankar arrived at by extremizing the energy. Analytic solutions are not known. We can however numerically solve these equations as will be discussed in section 4.4.

Comparing (4.2.8) with the equations for a Nielsen-Olesen vortex (2.4.14), we find that although the form of the equations is more or less similar, the Alice string structure is significantly different due to an extra component m, which, as will be shown in the next sections, turns out to be non-zero at the origin.

4.3 The asymptotic behaviour of the fields

We now determine the asymptotics of the fields. Let us first consider the behaviour of the fields close to the origin.

To ensure regularity at the origin, the fields α and q must smoothly go to zero as r vanishes; but m need not go to zero, since the m-term of the ansatz has no θ dependence (4.2.3a). So

$$\alpha \to 0, \qquad q \to 0, \qquad m \to c_3, \tag{4.3.1}$$

for $r \to 0$. Using the trial functions:

$$\alpha = c_1 r^{n_1}, \tag{4.3.2}$$
$$q = c_2 r^{n_2}, \tag{4.3.3}$$
$$m = c_3, \tag{4.3.4}$$

we find that near the origin the fields are, to leading order,

$$\alpha \approx c_1 r^2, \tag{4.3.5a}$$
$$q \approx c_2 r, \tag{4.3.5b}$$
$$m \approx c_3. \tag{4.3.5c}$$

For large r the Higgs and gauge fields must approach the topological stable form (4.1.17). Therefore, the boundary conditions for $r \to \infty$ are:

$$\alpha \to 1, \qquad q \to a, \qquad m \to a, \tag{4.3.6}$$

where a is the rescaled constant (4.1.8):

$$a_\pm = \frac{-\chi \pm \sqrt{\chi^2 + 24\xi^2}}{12\xi}. \tag{4.3.7}$$

To obtain the rate by which the fields approach their asymptotic values, we use the ansatz

$$\alpha \to 1 + \delta\alpha, \tag{4.3.8a}$$
$$q \to a + \delta q, \tag{4.3.8b}$$
$$m \to a + \delta m. \tag{4.3.8c}$$

We keep only lowest order terms in δq and δm, but all terms in $\delta\alpha$. The resulting equations are

$$\delta\alpha'' - \frac{1}{r}\delta\alpha' = 9a^2\delta\alpha, \tag{4.3.9a}$$
$$\delta q'' + \frac{1}{r}\delta q' = \frac{a\delta\alpha^2}{r^2} + (\xi(30a^2 - 2) + 2\chi a)\delta q + (6\xi a^2 + 2\chi a)\delta m, \tag{4.3.9b}$$
$$\delta m'' + \frac{1}{r}\delta m' = (18\xi a^2 + 6\chi a)\delta q + (\xi(18a^2 - 2) - 2\chi a)\delta m. \tag{4.3.9c}$$

Substituting the trial function

$$\delta\alpha = \mathrm{e}^{-\rho r} r^{\sigma} (c_1^{\alpha} + r^{-1} c_2^{\alpha}), \tag{4.3.10}$$

in (4.3.9a) and equating the coefficients of $\mathrm{e}^{-\rho r} r^{\sigma}$ and $\mathrm{e}^{-\rho r} r^{\sigma-1}$ to zero, we obtain

$$\delta\alpha \approx c_1^{\alpha} \mathrm{e}^{-3a\,r} r^{1/2}, \tag{4.3.11}$$

where c_1^{α} and c_2^{α} are constants.

To obtain the asymptotic form of δq and δm, we proceed in a similar way. Using the trial functions

$$\delta q = \mathrm{e}^{-\rho r} r^{\sigma} (c_1^{q} + r^{-1} c_2^{q}), \tag{4.3.12a}$$

$$\delta m = \mathrm{e}^{-\rho r} r^{\sigma} (c_1^{m} + r^{-1} c_2^{m}), \tag{4.3.12b}$$

we obtain the particular and the homogeneous solutions of (4.3.9b) and (4.3.9c). The calculation is more subtle than the previous one and we therefore amplify on relevant parts of the calculation.

Let us first consider only the homogeneous equations. Substituting (4.3.12) into (4.3.9b) and (4.3.9c) and equating the coefficients of $\mathrm{e}^{-\rho r} r^{\sigma}$ to zero, yields:

$$\rho^2 c_1^q = (\xi(30a^2 - 2) + 2\chi a) c_1^q + (6\xi a^2 + 2\chi a) c_1^m,$$
$$\rho^2 c_1^m = (18\xi a^2 + 6\chi a) c_1^q + (\xi(18a^2 - 2) - 2\chi a) c_1^m.$$

These equations have two solutions for ρ^2,

$$\rho_{\pm}^2 = 2\xi(12a^2 - 1) \pm 4a(3\xi a + \chi).$$

Substituting $a_{\pm}$ (4.3.7), we find

$$\rho_{+}(a_{\pm})^2 = 4\xi + \frac{\chi^2}{6\xi} \mp \frac{\chi\sqrt{\chi^2 + 24\xi^2}}{6\xi}, \tag{4.3.13a}$$

$$\rho_{-}(a_{\pm})^2 = \frac{\chi^2}{2\xi} \mp \frac{\chi\sqrt{\chi^2 + 24\xi^2}}{2\xi}. \tag{4.3.13b}$$

Since ρ^2 is invariant under the transformation

$$a_{-} \to -a_{+}, \qquad \chi \to -\chi, \qquad \xi \to \xi,$$

we can confine ourselves to a_{+}. In order that the field q and m fall off exponentially, both ρ_{+}^2 and ρ_{-}^2 must be positive. A little functional analysis shows this condition to be fulfilled if

$$\chi < 0 \qquad \text{and} \qquad \xi > 0. \tag{4.3.14}$$

The asymptotic behaviour is governed by the lower value of γ_+ and γ_-. So the exponent is:

$$\rho = \begin{cases} \sqrt{\rho_-^2} & \text{if } \chi > -\sqrt{3}\xi, \\ \sqrt{\rho_+^2} & \text{if } \chi < -\sqrt{3}\xi. \end{cases} \tag{4.3.15}$$

We can now, without much effort, determine the factor σ and the particular solution. Putting everything together, we finally get:

$$\delta q \approx \delta q_h + \delta q_p = \frac{c_1^q \mathrm{e}^{-\rho r}}{\sqrt{r}} + c_h^q(\chi, \xi) \frac{(c_1^\alpha)^2 \mathrm{e}^{-6ar}}{r}, \tag{4.3.16a}$$

$$\delta m \approx \delta m_h + \delta m_p = \frac{c_2^m \mathrm{e}^{-\rho r}}{\sqrt{r}} + c_h^m(\chi, \xi) \frac{(c_1^\alpha)^2 \mathrm{e}^{-6ar}}{r}, \tag{4.3.16b}$$

where c_h^q and c_h^m are complicated functions of ξ and χ.

4.4 Numerical solutions

We want to find solutions to the differential equations (4.2.8) that satisfy the boundary conditions at zero (4.3.1) and at infinity (4.3.6). This kind of problem is known as a two point boundary value problem. It can be solved by using a relaxation method. In this method the differential equations are replaced by finite-difference equations on a mesh of points that covers the range of interest. Then starting with a trial function consisting of values for the dependent variables at each mesh point, all the values are iteratively adjusted so as to bring them into successively closer agreement with the finite-difference equations, and simultaneously, with the boundary conditions. If the trial function was not too far off, the result will *relax* to the true solution. (A detailed treatment of this technique can be found in Press *et al.* (1992).)

We consider the solution we obtained where we have set $\xi = 1$ and $\chi = -1$. Figure 4.1 shows a plot of the functions $\alpha(r)$, $q(r)$ and $m(r)$.

This string solution features a T_1 magnetic field along the z-axis,

$$B_z = \frac{1}{2} \frac{\partial}{\partial r} \left(\frac{\alpha}{r} \right) T_1. \tag{4.4.1}$$

The r-dependence of this field is plotted in figure 4.2.

The energy of the solution is obtained by substituting the ansatz (4.2.3) in the generic energy functional (2.1.15). Thus we find that the energy per unit

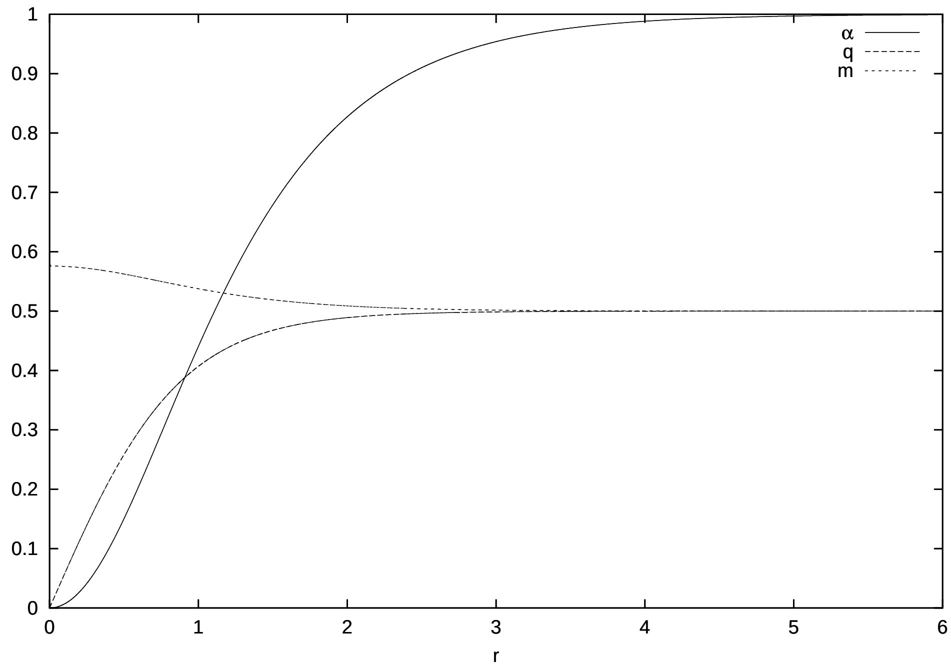


Figure 4.1: Functions $\alpha(r)$, $q(r)$ and $m(r)$ for the cylindrically symmetric string solution, with $\xi = 1$ and $\chi = -1$.

length of string equals

$$\begin{aligned} E = 2\pi \int \Bigg\{ & \frac{1}{8r^2}\left(\frac{d\alpha}{dr}\right)^2 + \frac{9q^2}{8r^2}(\alpha-1)^2 + \frac{3}{8}\left(\frac{dm}{dr}\right)^2 + \frac{9}{8}\left(\frac{dq}{dr}\right)^2 \\ & + \frac{1}{4}\xi\left[-(3m^2+9q^2) + \frac{1}{4}(3m^2+9q^2)^2\right] + \frac{1}{4}\chi(9mq^2 - m^3) + V_c(a)\Bigg\}\, rdr \\ = 2\pi \int & \rho_E(r)\, rdr, \end{aligned} \tag{4.4.2}$$

where we have added the constant

$$V_c(a) = \xi(3a^2 - 9a^4) - 2\chi a^3, \tag{4.4.3}$$

to the potential so that its minimum value is zero. Figure 4.3 shows the energy density of the solution. The energy per unit length of string is finite. Of course, the total energy of an infinitely long string is infinite. But, based on the straight-line strings, one can construct closed strings that have finite energy. Such Alice loops are only approximate solutions of the field lagrangian and they will collapse after a certain time since the topological argument only ensures local stability of the string.

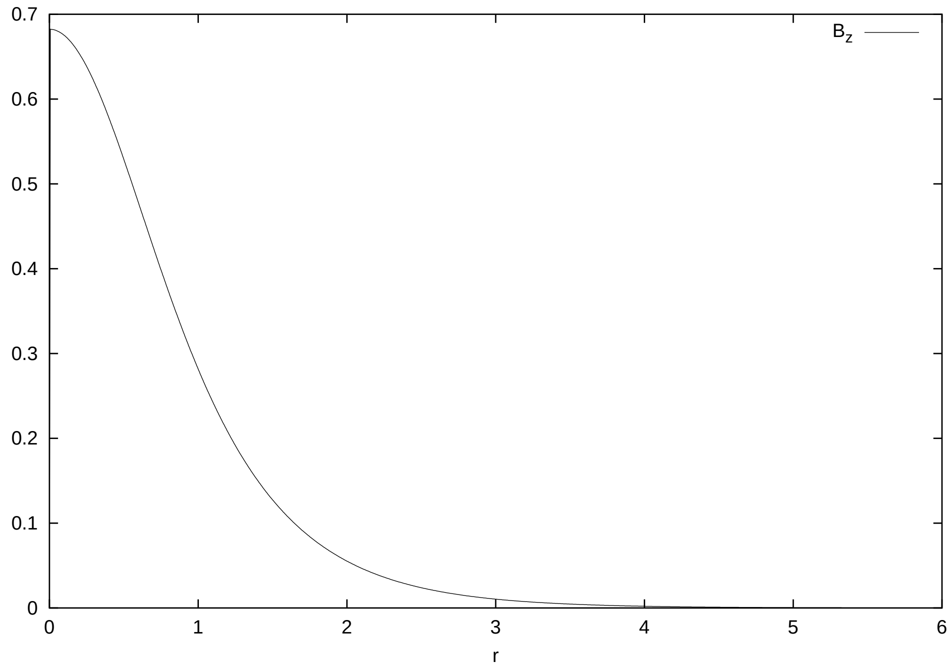


Figure 4.2: The magnetic field, B_z, for the string solutions with $\xi = 1$ and $\chi = -1$.

The figures illustrate that the string solution is well-localized, in spite of the fact that we do have an unbroken $U(1)$ symmetry. However, the $\mathbb{Z}_2$ flux is not associated with the unbroken generator T_3, but with a generator orthogonal to T_3, the broken generator T_1. The A^1 field is therefore massive and dies off exponentially.

In contrast with the Nielsen-Olesen vortex, the Alice string does *not* have a zero Higgs field at the origin. The value of $m(0)$ is not to be chosen at will, but appears to be enforced by the choice of the boundary values at infinity, implying that it is a function of the two parameters ξ and χ. We did not succeed in finding the explicit functional dependence.

Since the Higgs field is non-zero at the origin, the full $SO(3)$ symmetry is not restored in the string core. Taking the form

$$\Phi(r=0) = \frac{m}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.4.4}$$

the field at the origin is invariant under rotations about the T_1 axis and under rotations by 180° about axes that lie in the T_2T_3-plane. Thus the residual symmetry group for $r = 0$ is $U(1) \rtimes \mathbb{Z}_2$. For $r \to \infty$, the residual symmetry is also $U(1) \rtimes \mathbb{Z}_2$, but it is composed of different generators. For intermediate values of r, only a $\mathbb{Z}_2$ symmetry remains.

The fact that the Higgs field is non-zero at the origin suggests that the Alice

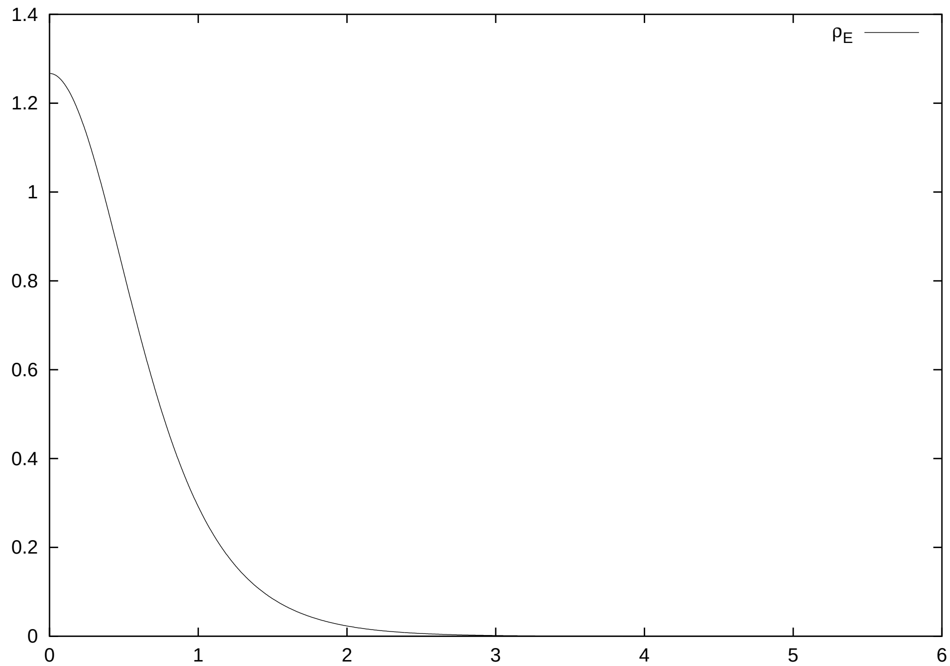


Figure 4.3: The energy density ρ_E for the string solutions with $\xi = 1$ and $\chi = -1$.

string can be turned into an electric superconductor, and indeed, we have been able to construct a superconducting Alice string solution. We present this solution in chapter 6, but to stress the importance of it, we first discuss, in the next chapter, the standard physical interpretation of charged Alice strings.

Chapter 5

Alice strings and non-localized charge

Let us consider an experiment similar to the one described in section 3.2. We take two charges of equal charge q and a closed loop of Alice string. The two charges are brought together and found to attract each other. If we now carry one of the charges through the Alice loop, it flips sign, and after reuniting the two charges, they repel. If we assume that the total charge is conserved, a puzzle presents itself: what happened to the $2q$ units of charge we seem to have lost?

The standard view contends that the missing charge cannot be transferred to the string itself, nor to a sheet spanned by the string loop. Nevertheless, charge *is* left behind. The loop has acquired a long-range electric field from which its charge can be inferred, but the source of this charge cannot be localized. This new type of ghostly, non-localizable charge, only noticeable by its long-range electric field, has been called "Cheshire charge" (Carroll, 1865).

In this chapter we spell out the arguments that lead to this view. Two important papers discussing Cheshire charge are Alford *et al.* (1991), who coined the term 'Cheshire charge', and Preskill and Krauss (1990). An especially clear and entertaining overview can be found in Preskill (1993). The following discussion is based on these three papers.

Although we do not deny the possibility of Cheshire charge, we do think that Cheshire charge does not exhaust the possible ways in which an Alice string can be charged. In the next chapter we will argue that under certain circumstances charge can also be localized on the string.

5.1 The electric field

Cheshire charge owes its existence to the peculiar boundary conditions of the electric field in the presence of an Alice loop. Let us therefore first analyse the characteristics of the electric field.

We can define the electric field at a point x by measuring the force that acts on a positive test charge located at x. To make this definition work, we obviously need a procedure to ascertain the sign of the test charge. A simple procedure is the following.

We take an arbitrary charge and call it positive by convention. This charge is for calibration and we will always keeps it at the same place. With the help of this calibration charge, we can establish the sign of any test charge by simply taking it to the calibration charge and seeing whether the test charge is attracted or repelled.

First suppose there are no Alice loops present, only charged particles. Then, when we want to know the electric field at x, we can pick a test charge, verify its sign, then move it to x and measure the force it experiences there. This way the electric field is defined unambiguously. In effect, the procedure fixes the gauge such that the charge operator Q points in the same direction everywhere (cf. figure 3.2).

However, if Alice loops are present, the procedure will fail. The result of calibrating a test charge is now ambiguous, since it depends on how many times the charge winds through a loop before it arrives at the calibration charge. Thus we cannot consistently fix the sign of the electric field everywhere.

Of course, if we take care that a test charge keeps out of the way of the Alice loops, the electric field can still be determined unambiguously in the area beyond the Alice loops. But we can extend this area to the whole space in the following way. We take a sheath, stretch it tightly across the string loop and demand that no test charge is to cross the sheath when it is sent out to measure the electric field. As long as the test charge does not pass the sheath we can be sure of its sign. With this procedure, the sign of the electric field is unambiguously defined, at the cost though of introducing unusual boundary conditions at the sheath. If the positive test charge were to go through the sheath, it would become a negative test charge. (After all, after taking it to the calibration charge in order to compare it, the test charge has in effect once circled the string.) But the sheath is just an artifact and not physically observable; so if the test charge passes the sheath, the force on the charge must not abruptly change. It follows that as the charge of the particle changes sign at the sheath, the electric field we measure must also change sign at the sheath. Summing up, we have fixed the sign of the electric field on the whole space by introducing a sheath at which the electric field changes sign and the charge of a particle flips.

We can restate the above in more abstract terms. In the presence of an Alice loop, the charge operator Q is necessarily double valued. No gauge transformation can remove this double-valuedness. But we can make a singular gauge transformation such that Q points in the same direction everywhere at the cost of introducing a discontinuity, or a branch cut, on a surface bounded by the string loop. At the cut, the electromagnetic field changes sign and the charge of a charged particle flips. As a result, we have picked out a single sheet of the two-valued operator Q, joined to the second sheet at a branch cut. The branch cut is just a gauge artifact and its location is therefore arbitrary. The cut is not physically observable; by convention we have decreed that a charge changes sign at the cut, but in reality, we cannot pinpoint where the change in sign takes place.

5.2 Cheshire charge

Let us now return to the gedanken experiment discussed in the introduction of this chapter. We suppose we have a region that contains a closed string and two point charges, such that the total charge is zero as measured on a distant boundary. The two point charges are united and found to be equal, but opposite in sign: $+q$ and $-q$. The charge inside the surface Σ that encloses the string loop is zero. This state of affairs is shown in figure 5.1a. Now, we take the particle with charge $+q$ through the loop while leaving the other behind. When the charges are reunited, both have charge $-q$ (see figure 5.1b). If we hold on to charge conservation, the total charge as measured on the distant boundary cannot have changed and we must assume $2q$ units of charge have been deposited inside Σ.

It is natural to think that the $2q$ charge has been transferred to the string core. But this does not seem to be possible, because it is thought that an unclosed Alice string has no charged excitations (see below). Moreover, it is unclear by which mechanism a particle could transmit charge to the string core. Yet, charge conservation requires that the string loop does carry a charge. So, apparently, the point particle is able to excite a charged state of the string loop without really transferring any charge to the string core. It is the two-valuedness of the electric field which makes this possible.

The charge transfer process is sketched in figure 5.2. The figure shows one branch of the two-valued electric field of a point charge q for a sequence of positions of the charge. The total electric charge as measured at a far boundary is assumed to be q. The initial electric field of the particle in the vicinity of a string loop is shown in 5.2a. When the charge reaches the cut, it disappears behind the cut just as its image charge $-q$ on the second sheet emerges from behind the cut (figure 5.2c and 5.2d). The electric flux emanating from the image charge returns to the second sheet through the cut, while the flux emanating from the original positive charge (now on the second sheet) returns to the first sheet through the cut. After the charge has passed the cut, an observer on a closed surface that encloses the Alice loop, but not the point charge, measures electric flux $4\pi q$ through the surface and infers that $2q$ units of charge are inside. The cut seems to have acquired this charge $2q$ (figure 5.2f). But in fact this charge is not localized anywhere. The branch cut that appears to be the source has no invariant significance. We can move the cut without really changing the physics

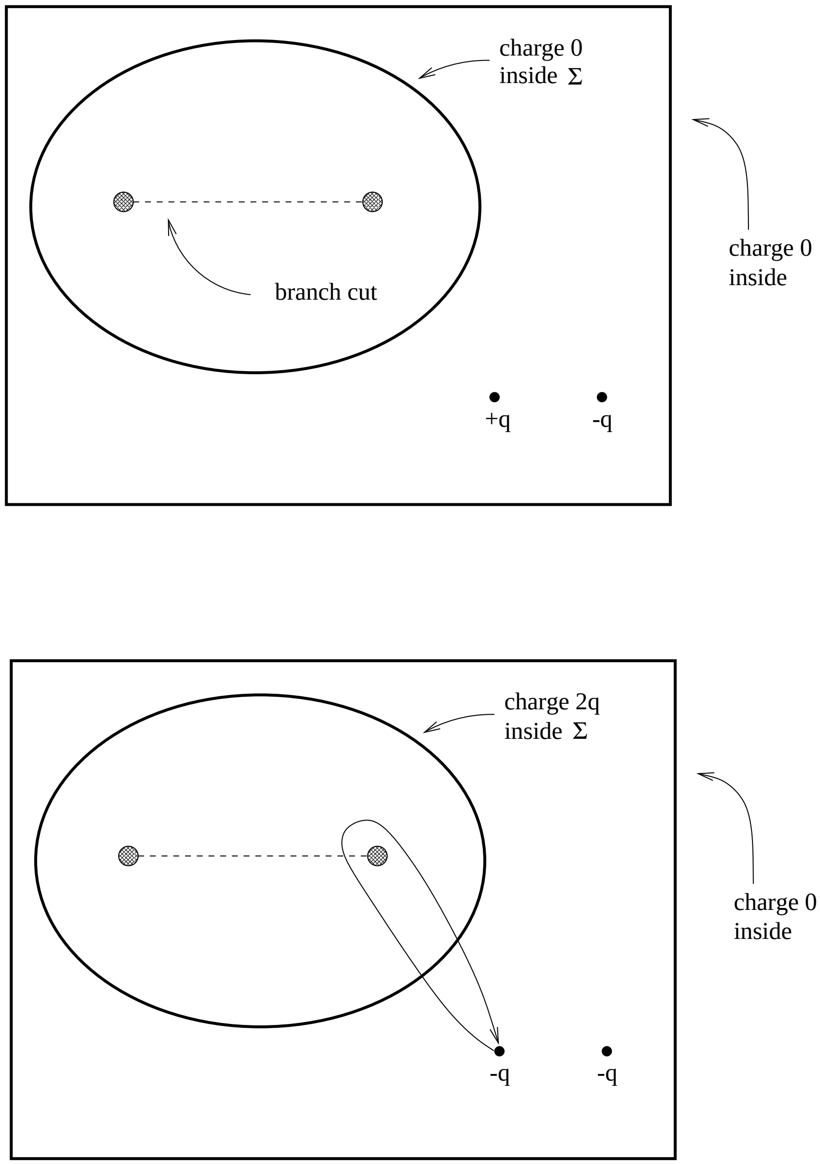


Figure 5.1: A region contains two charged particles and a string loop, of which only a cross section is shown. Initially, the particles have charge q and $-q$. The particle with charge $+q$ travels through the string loop. On returning, the sign of its charge has changed and $2q$ units of charge have been deposited inside Σ.

of the process. Consequently, the loop carries a charge that is not localizable. Such unlocalized charge has been named "Cheshire charge".

Let us try to make the above discussion a bit more precise. For this purpose, we rehash the arguments as given in Alford *et al.* (1991). First we analyze the electric field in the presence of a charged particle and a single, unclosed Alice string.

The $U(1)$ electric field E_i is defined in terms of the $SO(3)$ gauge field F_{0i},

$$F_{0i}(\boldsymbol{r}) = E_i(\boldsymbol{r}) S_3(\boldsymbol{r}). \tag{5.2.1}$$

Since S_3 is a double-valued function of θ, E_i must also be a double-valued function in order that the gauge field F_{0i} is single valued.[1]

Aligning the string along the z-axis, we can project the first sheet of the double cover onto the the xy-plane. This enables us to depict E_i as a single-valued function with a branch cut. We will project the $\theta \in [3\pi, 4\pi)$ segment of E_i onto the half-space $y < 0$, and the $\theta \in [0, \pi)$ segment onto $y > 0$, with a branch cut along the $y = 0, x < 0$ plane.

Placing a charge $+q$ at $x = R, y = z = 0$, we can now show that the electric field on the first sheet is the same as the one that would be obtained for single-valued E_i with conducting plate boundary conditions at the branch cut (figure 5.3). The full $E_i(\theta)$ (suppressing the r and z dependence) is defined over $\theta \in [0, 4\pi)$. It obeys the source-free Gauss' law,

$$\partial_i E_i = 0, \tag{5.2.2}$$

which follows from the lagrangian (4.1.1), and the boundary condition

$$E_i(\theta) = -E_i(\theta + 2\pi). \tag{5.2.3}$$

We place a charge $+q$ at $x = R$ in physical space. This implies imposing point charge boundary conditions for a charge $+q$ at $(r, \theta, z) = (R, 0, 0)$ and a charge $-q$ at $(R, 2\pi, 0)$. The configuration is symmetric under reflection in the $y = 0$ plane, i.e. under the transformation $\theta \to 4\pi - \theta$, and so we must have

$$E_{x,z}(\theta) = -E_{x,z}(4\pi - \theta), \qquad E_y(\theta) = E_y(4\pi - \theta). \tag{5.2.4}$$

Applying these conditions at $\theta = \pi, 3\pi$, we find

$$E_x(\pi) = E_z(\pi) = E_x(3\pi) = E_z(3\pi) = 0, \qquad E_y(\pi) = -E_y(3\pi). \tag{5.2.5}$$

This is exactly the "no parallel electric field" boundary condition for a conducting plate at the the branch cut, where we jump from $\theta = \pi$ to $\theta = 3\pi$.

[1] According to Alford *et al.*. No reasons are given why the gauge fields $F^a_{\mu\nu}$ must be single valued.

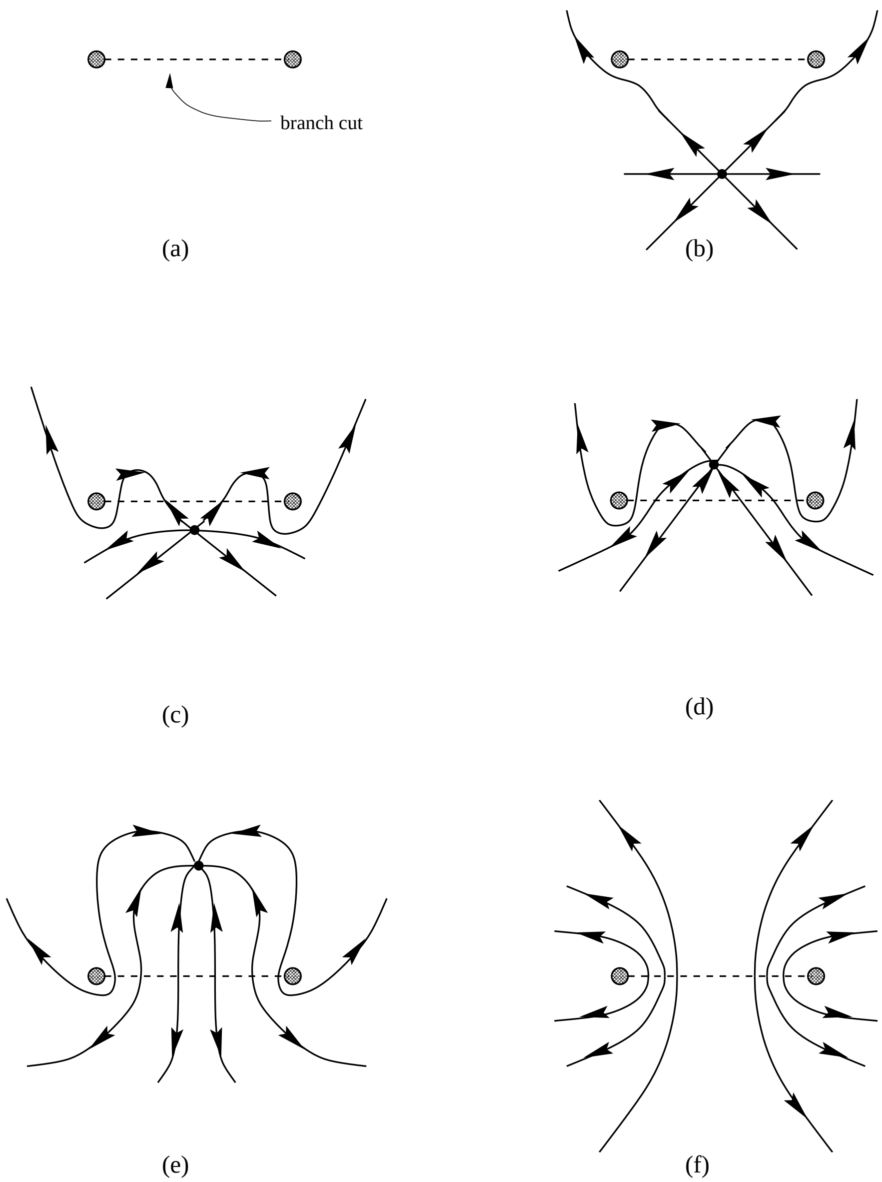


Figure 5.2: One branch of the electric field of a point charge in the vicinity of a string loop, for a sequence of positions of the point charge.

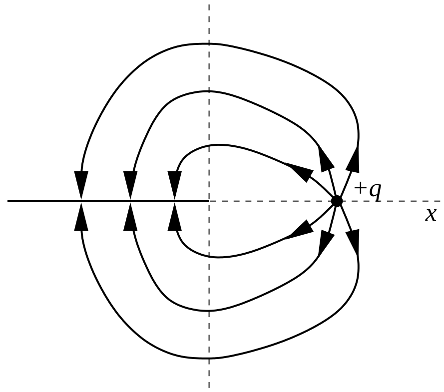


Figure 5.3: Electric field of a point charge in the presence of an Alice string. The string is aligned along the z-axis. Shown is the first sheet ($\theta = 0$ to π and 3π to 4π), with a branch cut on the negative x-axis.

The charge is attracted to the string, analogous to a charge that is a attracted to a conducting plate. A charge near a conducting plate induces an opposite charge on the plate, attracting the original charge. Although there is not any such induced charge at the branch cut, the electric field is the same. The energy of the field will be decreased if the charge moves closer to the string, so the charge is really attracted to the string.

Let us now consider a string loop carrying Cheshire charge. We represent the loop by two strings, one at $(x, y) = (0, 0)$ and another at $(D, 0)$, and let the plane joining the two strings be the branch cut. The electric field E_i must change sign on circling one string, but is single valued on circling both strings. For this reason, a single string cannot have a charged mode, but a system of two strings can. (This is a consequence of the above mentioned constraint that demands the gauge field $F^a_{\mu\nu}$ to be single valued. At this point, we cannot explain this. We return to it in the next chapter.)

Far from the strings, this charged mode manifests itself as a radial electric field,

$$E_i(r, \theta) = Q/r. \tag{5.2.6}$$

We cannot localize the source of this field. A surface enclosing no strings contains no net charge because there is no source. Surfaces enclosing a single string have no meaning because on them E_i is double-valued.

We can calculate $E_i(x, y)$ to obtain more information. It can be shown, by reflection symmetry in the x- and y-axes, that the branch cut is equivalent to a conducting plate with charge Q (figure 5.4). Solving the electrostatic equation (5.2.2) with this boundary condition shows that, in the limit that the string

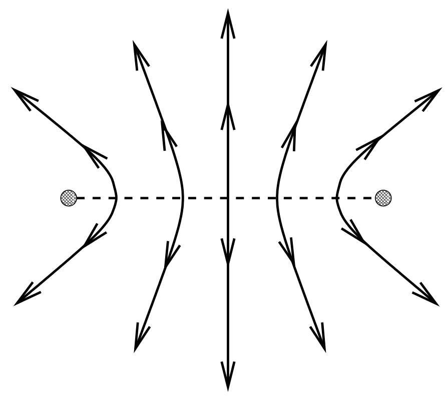


Figure 5.4: Electric field of a charged mode on an Alice string pair, first sheet.

separation D is much greater than the core size,

$$E_y(x,0) = q/\sqrt{x(D-x)}, \quad \text{for } 0 < x < D \tag{5.2.7}$$

If we take one string to infinity ($D \to \infty$) the electric field a fixed distance x from the other decreases towards zero, confirming that there is no charge localized near that string.

To conclude this chapter, we like to point out that charge quantisation (which necessary holds in this theory) forbids the electric field lines to cross the core of an Alice string. Hence, an Alice loop is able to acquire Cheshire charge by trapping electric flux.

Chapter 6

Alice strings carrying electric charge

As we saw, an Alice loop can carry Cheshire charge by trapping electric flux. The flux lines cannot cross the string and flow away since that would break charge conservation and quantisation. This type of flux confinement reminds one of the Meissner effect; it suggests that the electric flux is kept together by a *magnetic* superconducting current flowing through the string. This view is supported by the fact that in the present model a magnetic monopole can be smoothly deformed to a magnetically charged Alice loop (as was explained in chapter 3). So at least, a magnetically charged Alice loop does not meet a topological obstruction. From a magnetically charged Alice loop, it is a natural step to a magnetic superconducting loop. Indeed, it is commonly thought that if all components of the order parameter vanish at the core of a string and the non-abelian symmetry is fully restored there, the core will be turned into a magnetic superconductor. Thus by applying the appropriate boundary conditions, we might expect to find a magnetic superconducting Alice string.

In chapter 3 we noted that when a magnetic monopole is pulled through an Alice loop, magnetic charge is transferred to the loop region. Where can this charge go? Possibly, it could take the form of magnetic Cheshire charge, i.e. the loop could acquire a long-range magnetic field, without there being a localized source of magnetic charge on the loop or in its direct vicinity. In line with our suggestion about electric Cheshire charge, we could conceive magnetic Cheshire charge as an *electric* superconducting current that has trapped magnetic flux.

However, as we argued above, an Alice loop can also have magnetic charge localized at its core. So we are lead to conclude that an Alice loop can carry magnetic charge in two distinct ways: as magnetic Cheshire charge, with a tangential magnetic field generated by an electric supercurrent; or as magnetic charge localized at its core, producing a radial magnetic field.

Extending this line of thought, we expect there to be two similar alternatives for an Alice loop carrying electric charge: the loop could have electric Cheshire charge, with a tangential electric field generated by a magnetic supercurrent; or the loop could have electric charge localized at its core, producing a radial electric field.

If correct, this view would mean that in the original $SO(3)$ model broken down to an effective unbroken $U(1)$, the electric-magnetic dual symmetry (say the analogon of the Olive-Montonen symmetry) is a priori not broken by the Alice string, but may be broken by either an electric or magnetic charge excitation of the string core. We are suggesting that there are two alternative ways to achieve this, which are each others dual image. The $\mathbb{Z}_2$ flux itself has no $U(1)$ electric or magnetic content and it is by imposing suitable boundary conditions that we may obtain either $U(1)$ electric or $U(1)$ magnetic excitations of the string.

In this chapter and the next one we provide evidence in support of this view. The present chapter deals with Alice strings carrying electric charge. Specifically, we show that an Alice string can support localized electric charge and electric current if the appropriate boundary conditions are imposed. We use a general ansatz (section 6.1) and derive reduced field equations for string solutions with either localized electric charge, electric current, or both (section 6.2). We determine the behaviour of these solutions for large and small r (section 6.3) analytically, and for intermediate values by numerical calculations (section 6.4). Finally, we examine the gauge invariant physical characteristics of the solutions, showing them to be consistent (section 6.5).

In the next chapter we subsequently explore the possibility of an Alice string carrying magnetic charge.

6.1 The ansatz

Generally speaking, a string can be turned into a superconductor if electromagnetic gauge invariance is broken inside the string (Witten, 1985; Everett, 1988). This also applies to the Alice string. The Alice string solution discussed in chapter 4 takes the form

$$\Phi = m(r)\Phi_1 + q(r)\mathrm{e}^{T_1\theta/2}\Phi_2\mathrm{e}^{-T_1\theta/2}, \qquad (4.2.3a)$$

$$A_\theta = \frac{\alpha(r)}{2er}T_1. \qquad (4.2.3b)$$

Far from the core, the Higgs field is invariant under rotations generated by the charge operator $S_3(\theta)$, while at $r = 0$, it is invariant under rotations generated by T_1. A $S_3(\theta)$ transformation would rotate the core condensate, but leave the Higgs field at large distance invariant. To construct charged excitations, we let the condensate undergo a S_3 rotation in the z and t directions in such a way that the resulting configuration is still a solution to the field equations. This can be effected by making the "truncated gauge transformation"

$$\Phi = S\overline{\Phi}S^{-1}, \tag{6.1.1}$$

$$A_\mu = S\overline{A_\mu}S^{-1} + \delta^i_\mu \frac{1}{e}\partial_i S S^{-1}, \tag{6.1.2}$$

where $\overline{\Phi}$ and $\overline{A_\mu}$ are the fields of the unperturbed string; $S = S(\boldsymbol{x}, t) \in G$, but tending to an element of H at infinity; and the index i varies over x and y. Though (6.1.1)-(6.1.2) looks like a gauge transformation, it is not, but rather represents a true physical excitation of the system: A_0 and A_z remain zero while A_x and A_y are transformed. Applying a true gauge transformation S^{-1}, we get the alternative expressions

$$\Phi = \overline{\Phi}, \tag{6.1.3}$$

$$A_\mu = \overline{A_\mu} - \delta^\alpha_\mu \frac{1}{e} S^{-1}\partial_\alpha S, \tag{6.1.4}$$

where α ranges over z and t.

We assume that the transformation S is of the form (Alford *et al.*, 1991)

$$S = \exp(\eta(z,t)s(r,\theta)), \tag{6.1.5}$$

where $s(r,\theta)$ is Lie algebra-valued. With this choice for S, the field A_α may be rewritten exactly as

$$A_\alpha = -\frac{1}{e}S^{-1}\partial_\alpha S = -\frac{1}{e}s\partial_\alpha \eta. \tag{6.1.6}$$

The complete ansatz for a electrically-charged string then takes the form

$$\Phi = m(r)\Phi_1 + q(r)\mathrm{e}^{T_1\theta/2}\Phi_2\mathrm{e}^{-T_1\theta/2}, \tag{6.1.7a}$$

$$A_\theta = \frac{\alpha(r)}{2er}T_1, \tag{6.1.7b}$$

$$A_\alpha = -\frac{1}{e}s\partial_\alpha \eta, \tag{6.1.7c}$$

which is just the ansatz for the bare string supplemented with a A_z and a A_0 component for the gauge fields. The A_z and A_0 components are expected to generate a tangential magnetic field (corresponding to an electric current running along the string) and a radial electric field (corresponding to an electrically charged string).

6.2 Reduction of the field equations

The next step in our construction is to substitute the ansatz (6.1.7) in the field equations

$$\text{for } a=1,2 \qquad D^\mu D_\mu \Phi^{aa} - D^\mu D_\mu \Phi^{33} = -2\left(\frac{\partial V}{\partial \Phi^{aa}} - \frac{\partial V}{\partial \Phi^{33}}\right), \tag{4.2.4a}$$

$$\text{for } a \neq b \qquad D^\mu D_\mu \Phi^{ab} = -2\frac{\partial V}{\partial \Phi^{ab}}, \tag{4.2.4b}$$

$$D^\mu F_{\mu\nu} = -e((D_\mu \Phi)T^c\Phi)T^c. \tag{4.2.4c}$$

We analyse these equations one by one. Consider first (4.2.4c) for $\nu = \alpha$

$$D^i D_i A_\alpha = e^2 \operatorname{Tr}([A_\alpha, \Phi]T^c\Phi)T^c \tag{6.2.2}$$

Substituting (6.1.7c) and dividing out the common factor $\partial_\alpha \eta/e$ yields

$$D^i D_i s = e^2 \operatorname{Tr}([s, \Phi]T^c\Phi)T^c \tag{6.2.3}$$

To proceed further, it is advantageous to expand s in a conveniently chosen local basis. This basis is formed by the generators S_a obtained by parallel transporting T_a in the background of the pure $\mathbb{Z}_2$ flux

$$S_a(\theta) = U(\theta)T_a U(\theta)^{-1} = \mathrm{e}^{T_1\theta/2} T_a \mathrm{e}^{-T_1\theta/2}. \tag{6.2.4}$$

Note that the generators $S_2(\theta)$ and $S_3(\theta)$ are double valued. The generator $S_3(\theta)$ is the charge operator $Q(\theta)$ used before (see chapter 3, particularly page 24). We may now write s as

$$s = p_a(r,\theta)S_a(\theta). \tag{6.2.5}$$

Substituting this in (6.2.3) yields three partial differential equations for the functions $p_a(r,\theta)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p_1}{\partial \theta^2} = 9e^2q^2p_1, \tag{6.2.6}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_2}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p_2}{\partial \theta^2} - \frac{p_2}{4r^2}(1-\alpha)^2 - \frac{1}{r^2}\frac{\partial p_3}{\partial \theta}(1-\alpha) = \frac{9}{4}e^2(m+q)^2p_2, \tag{6.2.7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_3}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p_3}{\partial \theta^2} - \frac{p_3}{4r^2}(1-\alpha)^2 + \frac{1}{r^2}\frac{\partial p_2}{\partial \theta}(1-\alpha) = \frac{9}{4}e^2(m-q)^2p_3. \tag{6.2.8}$$

Next consider (4.2.4c) for $\nu = r$, which gives

$$\partial^\alpha \partial_\alpha \eta \partial_r s + \partial^a \eta \partial_\alpha \eta [s, \partial_r s] = 0. \tag{6.2.9}$$

Substituting the expanded s (6.2.5) results in the following three equations

$$\partial^\alpha\partial_\alpha\eta\frac{\partial p_1}{\partial r} + \partial^\alpha\eta\partial_\alpha\eta\left(p_2\frac{\partial p_3}{\partial r} - p_3\frac{\partial p_2}{\partial r}\right) = 0, \tag{6.2.10}$$

$$\partial^\alpha\partial_\alpha\eta\frac{\partial p_2}{\partial r} + \partial^\alpha\eta\partial_\alpha\eta\left(p_3\frac{\partial p_1}{\partial r} - p_1\frac{\partial p_3}{\partial r}\right) = 0, \tag{6.2.11}$$

$$\partial^\alpha\partial_\alpha\eta\frac{\partial p_3}{\partial r} + \partial^\alpha\eta\partial_\alpha\eta\left(p_1\frac{\partial p_2}{\partial r} - p_2\frac{\partial p_1}{\partial r}\right) = 0. \tag{6.2.12}$$

And finally, for (4.2.4c) with $\nu = \theta$ we obtain similarly

$$\begin{aligned}\partial^\alpha\partial_\alpha\eta\frac{\partial p_1}{\partial \theta} + \partial^\alpha\eta\partial_\alpha\eta\left(p_2\frac{\partial p_3}{\partial \theta} - p_3\frac{\partial p_2}{\partial \theta} + \frac{p_2^2+p_3^2}{2}(1-\alpha)\right) + \\ r\frac{d}{dr}\left(\frac{1}{2r}\frac{d\alpha}{dr}\right) = e^2\frac{9q^2}{2}(\alpha-1),\end{aligned} \tag{6.2.13}$$

$$\partial^\alpha\partial_\alpha\eta\left(\frac{p_2}{\partial\theta} + \frac{p_3}{2}(\alpha-1)\right) + \partial^\alpha\eta\partial_\alpha\eta\left(p_3\frac{\partial p_1}{\partial \theta} - p_1\frac{\partial p_3}{\partial \theta} - \frac{p_1p_2}{2}(1-\alpha)\right) = 0, \tag{6.2.14}$$

$$\partial^\alpha\partial_\alpha\eta\left(\frac{p_3}{\partial\theta} - \frac{p_2}{2}(\alpha-1)\right) + \partial^\alpha\eta\partial_\alpha\eta\left(p_1\frac{\partial p_2}{\partial \theta} - p_2\frac{\partial p_1}{\partial \theta} - \frac{p_1p_3}{2}(1-\alpha)\right) = 0. \tag{6.2.15}$$

The remaining two equations, (4.2.4b) and (4.2.4c), yield respectively,

$$\begin{aligned}\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} = \partial^\alpha\eta\partial_\alpha\eta\left(4p_1^2q + (m+q)p_2^2 + (q-m)p_3^2\right) \\ + \frac{q(\alpha-1)^2}{r^2} - 2\mu^2q + 2mq\gamma + \lambda(9q^2+3m^2)q,\end{aligned} \tag{6.2.16}$$

$$\begin{aligned}\frac{d^2m}{dr^2} + \frac{1}{r}\frac{dm}{dr} = \partial^\alpha\eta\partial_\alpha\eta\left(4p_1^2q + 4(m+q)p_2^2 - 2(q-m)p_3^2\right) \\ -2\mu^2m + \gamma(3q^2-m^2) + \lambda(9q^2+3m^2)m.\end{aligned} \tag{6.2.17}$$

So, by substituting the ansatz (6.1.7), we arrive at eleven partial differential equations for the seven functions α, q, m, η and p_a. Although these equations look complicated they can be solved by making three reasonable assumptions.

1. For η we will consider three cases
   - $\eta = c_z z$, implying that $\partial^\alpha\partial_\alpha\eta = 0$ and $\partial^\alpha\eta\partial_\alpha\eta = -c_z^2$;
   - $\eta = c_t t$, implying that $\partial^\alpha\partial_\alpha\eta = 0$ and $\partial^\alpha\eta\partial_\alpha\eta = c_t^2$;

- $\eta \propto f((z \pm t)/\lambda)$, implying that $\partial^\alpha \partial_\alpha \eta = \partial^\alpha \eta \partial_\alpha \eta = 0$;

where c_z, c_t and λ are constants. We can cover all three cases by setting

$$\partial^\alpha \partial_\alpha \eta = 0, \tag{6.2.18}$$

$$\partial^\alpha \eta \partial_\alpha \eta = c_\eta, \tag{6.2.19}$$

with c_η a constant that can be either negative, positive or zero.

2. We can with impunity set p_1 to zero.
3. We let p_3 be a function of r only. This allows us to set p_2 to zero.

With these three assumptions the equations (6.2.6, 6.2.7, 6.2.10-6.2.12, 6.2.14, 6.2.15) are trivially satisfied which shows that the assumptions are consistent with the field equations. The remaining four equations take the form

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} = c_\eta p_3^2(\alpha - 1) + 9e^2q^2(\alpha - 1), \tag{6.2.20a}$$

$$\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} = c_\eta(q - m)p_3^2 + \frac{q(\alpha - 1)^2}{r^2} - 2\mu^2 q + 2mq\gamma + \lambda(9q^2 + 3m^2)q, \tag{6.2.20b}$$

$$\frac{d^2m}{dr^2} + \frac{1}{r}\frac{dm}{dr} = -2c_\eta(q - m)p_3^2 - 2\mu^2 m + \gamma(3q^2 - m^2) + \lambda(9q^2 + 3m^2)m, \tag{6.2.20c}$$

$$\frac{d^2p_3}{dr^2} + \frac{1}{r}\frac{dp_3}{dr} = \frac{p_3}{4r^2}(1 - \alpha)^2 + \frac{9}{4}e^2(m - q)^2 p_3. \tag{6.2.20d}$$

At this point it is convenient to rescale the fields and the r-coordinate (like we did in chapter 4, eq. (4.2.6)),

$$q \to \frac{\sqrt{\lambda}}{\mu}q, \qquad m \to \frac{\sqrt{\lambda}}{\mu}m, \qquad A_\mu \to \frac{\sqrt{\lambda}}{\mu}A_\mu, \qquad r \to \frac{e\mu}{\sqrt{\lambda}}r, \qquad c_\eta \to \frac{\lambda}{e^2\mu^2}c_\eta, \tag{6.2.21}$$

thereby making these quantities dimensionless. We define two new parameters:

$$\xi = \frac{\lambda}{e^2}, \qquad \chi = \frac{\gamma\sqrt{\lambda}}{\mu e^2}. \tag{6.2.22}$$

Finally then, we arrive at the dimensionless equations

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} = c_\eta p_3^2(\alpha - 1) + 9q^2(\alpha - 1), \tag{6.2.23a}$$

$$\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} = c_\eta(q - m)p_3^2 + \frac{q(\alpha - 1)^2}{r^2} + \xi(9q^2 + 3m^2 - 2)q + 2\chi mq, \tag{6.2.23b}$$

$$\frac{d^2m}{dr^2} + \frac{1}{r}\frac{dm}{dr} = -2c_\eta(q - m)p_3^2 + \xi(9q^2 + 3m^2 - 2)m + \chi(3q^2 - m^2), \tag{6.2.23c}$$

$$\frac{d^2p_3}{dr^2} + \frac{1}{r}\frac{dp_3}{dr} = \frac{p_3}{4r^2}(1 - \alpha)^2 + \frac{9}{4}(m - q)^2 p_3. \tag{6.2.23d}$$

Note that if $\eta \propto f((z \pm t)/\lambda)$, $c_\eta = 0$, then the equations for α, q and m are reduced to the equations for the bare string (4.2.8), solved in chapter 4. So, for this solution the T_1 flux and the Higgs field Φ are unaffected by the added A_0 and A_z gauge component and only the equation for p_3 remains to be solved.

For the solutions where $\eta = c_z z$ or $\eta = c_t t$, $c_\eta \neq 0$ and the three functions α, q, m are dependent on p_3. Thus, in the interior of the string the T_1 flux and the Higgs field Φ must change in order to support a long-range A_0 or A_z gauge component.

6.3 The asymptotic behaviour of the fields

As a first step in explicitly solving the differential equations for α, q, m and p_3 (6.2.23), we determine the asymptotic behaviour of these four functions. There are two basic configurations to consider: the one where $c_\eta = 0$ and those where $c_\eta \neq 0$. We start with the former.

If $c_\eta = 0$, the equations for α, q and m are reduced to the equations for the bare string (4.2.8). These equations were solved in chapter 4 and so here we need only to look at the equation for p_3. For $r \to \infty$, this equation takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_3}{\partial r}\right) = 0, \tag{6.3.1}$$

if we assume that α, q and m fall off faster than p_3. This assumption is justified by the solution of (6.3.1)

$$p_3 = c_1^{p_3} \log r + c_2^{p_3}. \tag{6.3.2}$$

This shows that p_3 has the right asymptotics; since

$$F^3_{\alpha r} \propto \partial_\alpha \eta \frac{\partial p_3}{\partial r} \propto \partial_\alpha \eta \frac{c_1^{p_3}}{r}, \tag{6.3.3}$$

this configuration has the radial electric field and tangential magnetic field that goes with an electric charge and current carrying string.

For $A_\alpha \propto p_3 S_3(\theta)$ to be regular, p_3 must approach zero as $r \to 0$. Substituting the asymptotic terms of α, q and m for small r (4.3.5) in (6.2.23d), we find that

$$p_3 \approx c_4 r^{1/2} \tag{6.3.4}$$

as $r \to 0$.

For the configurations where $c_\eta \neq 0$, the differential equations are completely coupled, thus affecting the asymptotics of α, q and m. First, consider the behaviour of the functions near the origin. To ensure regularity, the functions α, q and p_3

must go smoothly to zero as r vanishes; the function m need not go to zero, since the m-term of the ansatz has no θ dependence. Assuming the functions develop at the core as,

$$\alpha = c_1 r^{n_1}, \tag{6.3.5}$$
$$q = c_2 r^{n_2}, \tag{6.3.6}$$
$$m = c_3 r^{n_3}, \tag{6.3.7}$$
$$p_3 = c_4 r^{n_4}, \tag{6.3.8}$$

we find by substitution that

$$\alpha \approx c_1 r^2, \tag{6.3.9}$$
$$q \approx c_2 r^3, \tag{6.3.10}$$
$$m \approx c_3, \tag{6.3.11}$$
$$p_3 \approx c_4 r^{1/2}. \tag{6.3.12}$$

For large r the functions α, q and m must approach the the topological stable form (4.1.17) and so tend to the boundary values

$$\alpha \to 1, \qquad q \to a, \qquad m \to a. \tag{6.3.13}$$

(The charge or current carrying string is of course still an Alice string.) Assuming α, q and m fall of faster than p_3, we again find that p_3 evolves as the logarithm of r

$$p_3 \approx c_1^{p_3} + c_2^{p_3} \log r. \tag{6.3.14}$$

These field configurations have a long-range electromagnetic field

$$F^3_{\alpha r} \propto \partial_\alpha \eta \frac{\partial p_3}{\partial r} \propto \partial_\alpha \eta \frac{c_1^{p_3}}{r}. \tag{6.3.15}$$

For $\eta = c_z z$, the string configuration only carries an electric current that generates a long-range tangential magnetic field, $F^3_{zr} \propto c_z \frac{c_1^{p_3}}{r}$. For $\eta = c_t t$, the string configuration only carries an electric charge and a long-range radial electric field, $F^3_{zr} \propto c_z \frac{c_1^{p_3}}{r}$.

To obtain the asymptotic equations for α, q and m, we substitute

$$\alpha \to 1 + \delta\alpha, \tag{6.3.16a}$$
$$q \to a + \delta q, \tag{6.3.16b}$$
$$m \to a + \delta m, \tag{6.3.16c}$$

in (6.2.23a)-(6.2.23c) and keep only the highest order terms

$$\delta\alpha'' - \frac{1}{r}\delta\alpha'' = c_\eta (c_2^{p_3})^2 \delta\alpha \log^2 r, \tag{6.3.17a}$$

$$\delta q'' + \frac{1}{r}\delta q' = c_\eta (c_2^{p_3})^2 (\delta q - \delta m) \log^2 r, \tag{6.3.17b}$$

$$\delta m'' + \frac{1}{r}\delta m' = -2 c_\eta (c_2^{p_3})^2 (\delta q - \delta m) \log^2 r. \tag{6.3.17c}$$

As yet, we have not explicitly solved these equations.

6.4 Numerical solutions

To obtain solutions for finite r, we must solve the four differential equations (6.2.23) numerically. This can be achieved by solving the boundary value problem with a relaxation technique. (See section 4.4 for a short description of this technique.)

Again, the discussion of the results is divided into two parts. First we consider the field configurations for which $c_\eta = 0$, and then the configurations for which $c_\eta \neq 0$.

For $c_\eta = 0$ the differential equations for α, q and m can be solved independently of the equation for p_3. Their solutions were discussed in section 4.4. Knowing these solutions, we can solve the new function p_3 without difficulty. Figure 6.1 shows a solution for $\xi = 1$, $\chi = -1$, and $c_\eta = 0$.

We obtain the energy of this solution by substituting the ansatz (6.1.7) in the generic energy functional (2.1.15). Isolating the part of the energy contributed by p_3, the extra energy per unit length of the string is

$$\begin{aligned} \Delta E &= E_{tot} - E_{bare} \\ &= 2\pi \int \frac{1}{2}\left[(\partial_0\eta)^2 + (\partial_z\eta)^2\right]\left[\left(\frac{\partial p_3}{\partial r}\right)^2 + \frac{1}{r^2}\left(p_3 \frac{\alpha - 1}{2}\right)^2 + \frac{9}{4}(m-q)^2 p_3^2\right] r dr \\ &= 2\pi \int \rho_{\Delta E}(r)\, r dr, \end{aligned} \tag{6.4.1}$$

where E_{bare} is the energy of the "bare string" per unit length of the string (4.4.2). In figure 6.2 we have plotted the energy density multiplied by the factor r, for convenience's sake assuming that $\partial_0\eta = \partial_z\eta = 1$. Note that although the energy density $\rho_{\Delta E}(r)$ diverges at the origin,

$$\rho_{\Delta E(r)} \to r^{-1} \qquad \text{for } r \to 0, \tag{6.4.2}$$

the energy contribution at the core is nevertheless finite because of the volume factor rdr. The energy does however logarithmically diverge as r goes to infinity.

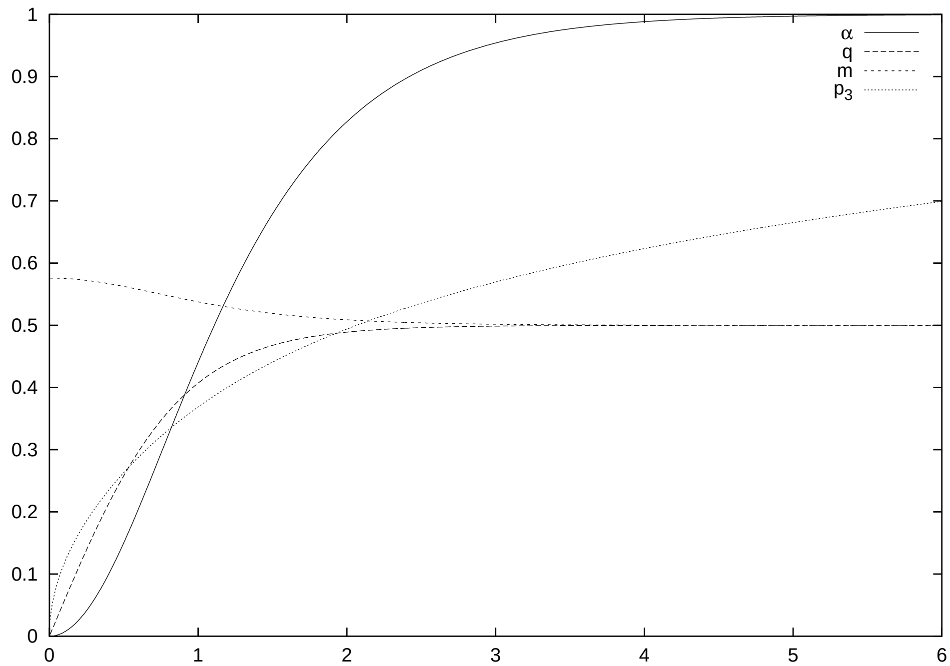


Figure 6.1: Functions $\alpha(r)$, $q(r)$, $m(r)$ and p_3 for the string solution carrying electric charge and current, with $\xi = 1$, $\chi = -1$, and $c_\eta = 0$.

This is obviously a consequence of the string being infinitely long and so it is an artifact that need not worry us.

We have arrived at explicit solutions that carry localizable electric charge and current. All the fields are regular and the energy density is finite everywhere. Still, an important question remains to be settled: are the solutions physically consistent in spite of the double-valuedness of some of the gauge fields? In the next section, this question will be answered in the affirmative.

The numerical program unfortunately failed to find solutions for configurations with $c_\eta \neq 0$. Several explanations are possible: the numerical program is not sophisticated enough; we used trial functions that are too far off from the "right solutions"; the program or the differential equations contain errors; or of course, solutions of the form (6.1.7) with $c_\eta \neq 0$ simply do not exist. Further investigation is needed to sort this out. Therefore, we stop at this point and focus on the solution with $c_\eta = 0$ in the remainder of this chapter.

6.5 The gauge-invariant characteristics

In this section we look at the gauge invariant characteristics of the new charged string solutions. These characteristics help us uncover the physical content of

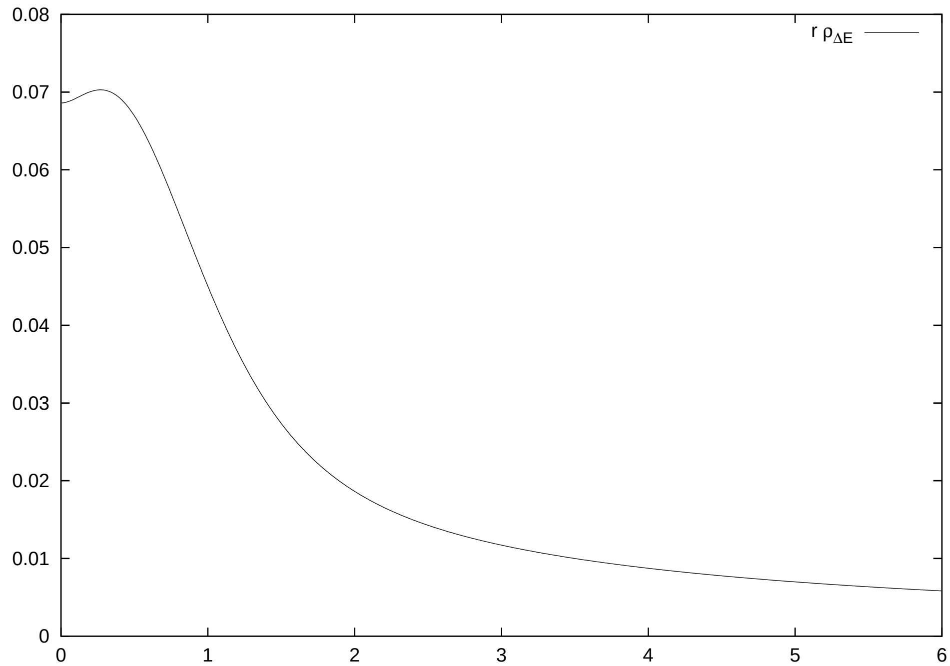


Figure 6.2: The energy density multiplied by r, $r\rho_{\Delta E}$, for the string solution carrying electric charge and current, with $\xi = 1$, $\chi = -1$, $c_\eta = 0$, and $\partial_0\eta = \partial_z\eta = 1$.

the solutions, since naturally, physically measurable quantities must always be gauge-invariant.

To construct gauge-invariant quantities we need to properly combine the Higgs field with the gauge fields. 't Hooft has done this for a $SU(2)$-gauge theory, defining the gauge-invariant electromagnetic field as (cf. section 2.5)

$$\mathcal{F}_{\mu\nu} = \frac{\phi^a}{\|\boldsymbol{\phi}\|} F^a_{\mu\nu} + \frac{1}{e\|\boldsymbol{\phi}\|^3}\varepsilon_{abc}\phi^a(D_\mu\phi^b)(D_\nu\phi^c), \tag{6.5.1}$$

This definition was devised for a Higgs field in the three dimension representation of $SU(2)$. Although our model contains a five dimensional Higgs field, we can make use of 't Hooft's formula, since the five dimensional Higgs field is expressible as the symmetric product of two three dimensional iso-vectors (see page 31):

$$\Phi^{ab} = \phi_1^a\phi_2^b + \phi_2^a\phi_1^b - \tfrac{2}{3}\delta^{ab}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2). \tag{6.5.2}$$

The two iso-vectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_1$ transform under a gauge transformation $S \in SO(3)$ as

$$\boldsymbol{\phi}_1 \to S\boldsymbol{\phi}_1, \tag{6.5.3a}$$
$$\boldsymbol{\phi}_2 \to S\boldsymbol{\phi}_2. \tag{6.5.3b}$$

So, by splitting up the Higgs field $\boldsymbol{\Phi}$ in two iso-vectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ we can exploit 't Hooft's formula (6.5.1) to obtain two gauge-invariant fields.

According to the ansatz (6.1.7), the Higgs field takes the form

$$\Phi = \mathrm{e}^{\theta T_1/2} \begin{pmatrix} m & 0 & 0 \\ 0 & -\frac{1}{2}m + \frac{3}{2}q & 0 \\ 0 & 0 & -\frac{1}{2}m - \frac{3}{2}q \end{pmatrix} \mathrm{e}^{-\theta T_1/2}, \tag{6.5.4}$$

which is equivalently expressed as

$$\boldsymbol{\phi_1} = \mathrm{e}^{\theta T_1/2} \frac{\sqrt{3}}{2} \begin{pmatrix} \sqrt{m-q} \\ 0 \\ \sqrt{2q} \end{pmatrix}, \tag{6.5.5a}$$

$$\boldsymbol{\phi_2} = \mathrm{e}^{\theta T_1/2} \frac{\sqrt{3}}{2} \begin{pmatrix} \sqrt{m-q} \\ 0 \\ -\sqrt{2q} \end{pmatrix} \tag{6.5.5b}$$

The two iso-vectors are anti-parallel at $r \to \infty$, defining the unbroken T_3-component and parallel at the origin, where T_1 is unbroken. In between these two extremes, the alignment is disrupted and only the $\mathbb{Z}_2$ symmetry remains.

Instead of directly using ϕ_1 and ϕ_2, it is more convenient to define the gauge-invariant fields with respect to the following linear combinations of ϕ_1 and ϕ_2

$$\varphi_1 = \phi_1 + \phi_2 \propto \mathrm{e}^{\theta T_1/2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6.5.6a}$$

$$\varphi_2 = \phi_1 - \phi_2 \propto \mathrm{e}^{\theta T_1/2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6.5.6b}$$

Inserting the ansatz (6.1.7) and iso-vector φ_1 respectively iso-vector φ_2 in the formula (6.5.1), we finally obtain two gauge invariant fields,

$$\mathcal{F}^1_{r\theta} = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\alpha}{r}\right), \tag{6.5.7}$$

$$\mathcal{F}^2_{\alpha r} = \partial_\alpha \eta \frac{\partial p_3}{\partial r}. \tag{6.5.8}$$

The field $\mathcal{F}^1$ contains the T_1 magnetic flux flowing along the string in the z-direction. The field $\mathcal{F}^2$ contains the long-range electromagnetic field associated with the charge generator S_3. Clearly, both these fields are single valued, just as it should. We are lead to conclude that our solutions are indeed physically acceptable.

For the field configuration with $c_\eta = 0$, we have plotted the field $\mathcal{F}^2$, assuming for convenience's sake that $\partial_\alpha \eta = 1$ (figure 6.3). The field $\mathcal{F}^2$ diverges at

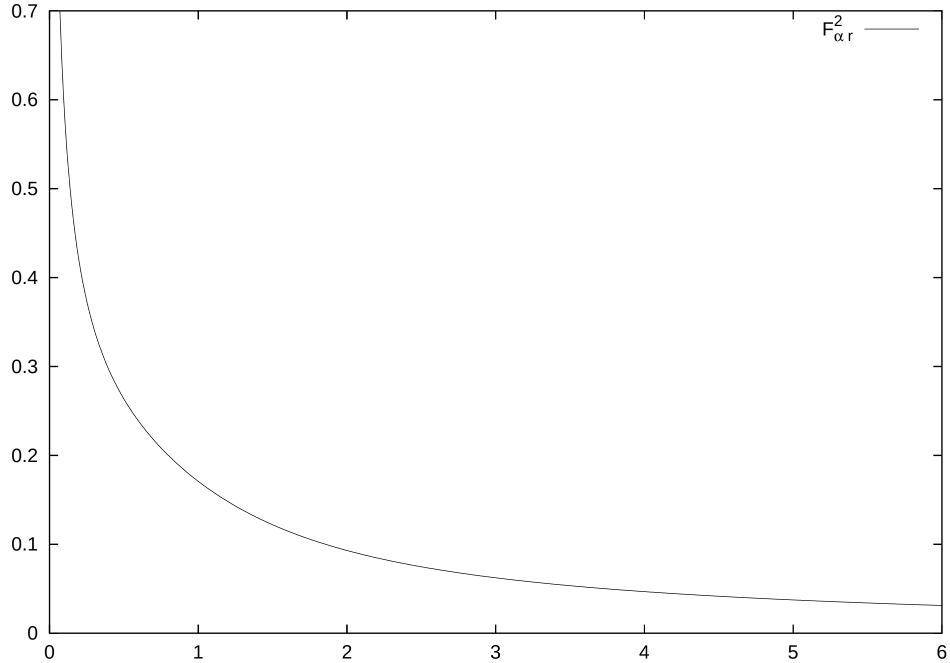


Figure 6.3: The gauge invariant field $\mathcal{F}^2_{\alpha r}$ for the string solutions with $\xi = 1$, $\chi = -1$ and $c_\eta = 0$.

the origin, but fortunately this behaviour is innocent as it is caused by the normalization factor $\|\varphi\|$ going to zero there. The quantity $\varphi^a F^a_{\mu\nu}$ assumes a finite, non-zero value at $r = 0$.

The orientation of the fields $\mathcal{F}^2_{0r}$, the electric radial field, and $\mathcal{F}^2_{zr}$, the magnetic tangential field, is sketched in figure 6.4.

To conclude this chapter, we note that the expressions (6.1.1), (6.1.2), (6.1.5) were adopted from Alford, Benson, Coleman, March-Russel, and Wilczek (1991). In their article, Alford *et al.* apply the transformation to a general class of non-abelian strings. They find solutions that correspond to zero modes of non-abelian strings. A pair of Alice strings has charged zero modes, but, they contend, a single Alice string does *not* have charged zero modes. Their argument is deceptively simple. The gauge fields A_μ (or equivalently, the fields $F_{\mu\nu}$) must be single valued, it is assumed, and this condition restricts the set of possible boundary conditions and thus precludes a single charged string solution. The premise, the single valuedness of the gauge fields, is not further explained, and indeed, one might question the necessity of this requirement. What must be single valued are of course the *gauge-invariant* quantities. The solutions presented above have gauge fields that are double valued; nonetheless their gauge-invariant quantities are single valued.

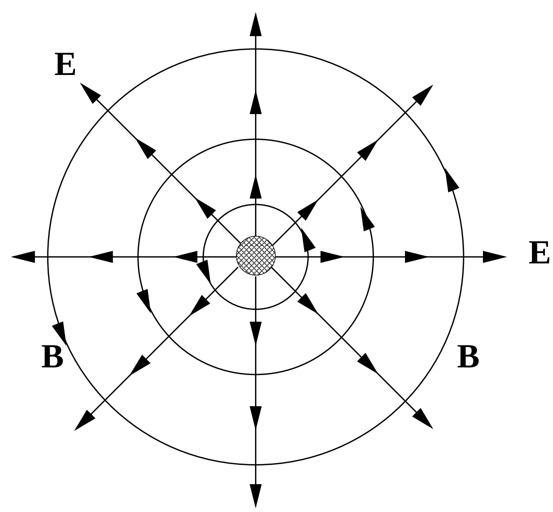


Figure 6.4: The gauge invariant electric- and magnetic fields of an Alice string that carries electric charge and current.

Chapter 7

Magnetically charged Alice strings

Alice strings can carry magnetic charge. If a magnetic monopole goes through an Alice loop, the monopole is turned into an anti-monopole and magnetic charge is transferred to the loop. In this chapter we discuss Alice strings that carry magnetic charge.

In the first section we define the magnetic charge carried by a string loop using a closed surface bounded to an arbitrary basepoint. In general, the surface can be bounded in many topologically inequivalent ways and the enclosed magnetic charge depends on this choice.

If a monopole goes through an Alice loop, the surfaces enclosing the monopole and the loop are deformed to new, topologically inequivalent surfaces. Thus we find that the charges of the loop and monopole have been changed by this process. We analyse this in section two.

In the third section, we present an ansatz for a string solution that carries magnetic charge and magnetic current.

The first two sections are based on Bucher *et al.* (1992).

7.1 A topological definition of magnetic charge

An Alice string carries a magnetic Z_2 flux flowing along the string. This flux, we recall, is defined by a closed path C that starts and ends at a basepoint x_0 and encircles the string exactly once (see figure 7.1),

$$h(C, x_0) = P \exp \left(\int_{(C,x_0)} A_i \, dx^i \right) \tag{7.1.1}$$

The flux h takes values in $H_d(x_0)$, the component of the stabilizor group of Φ at x_0 that is not connected to the identity. (Cf. chapter 3.) The value of the flux h depends, among other things, on the course of the path C. If we continuously

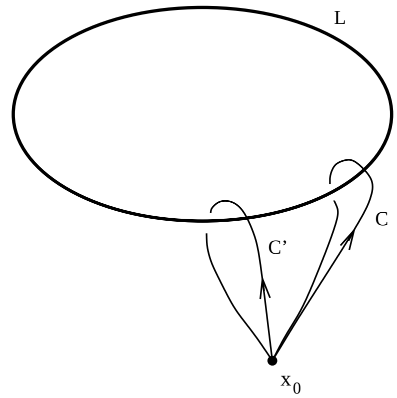


Figure 7.1: Two closed loops C and C' that each once encircle the loop of string L.

deform the path C to C', the associated flux continuously varies in $H_d(x_0)$ from $h(C, x_0)$ to $h(C', x_0)$.

Consider the family of closed paths based at x_0, $\{C_\phi \,|\, 0 \leq \phi < 2\pi\}$ (figure 7.2). As the parameter ϕ goes from 0 to 2π, the paths sweep out a degenerate torus enclosing the loop. Each path C_ϕ is associated with an element $h(C_\phi, x_0)$ of $H_d(x_0)$. In this way, from the family of paths, we obtain a closed path in H_d. The precise course of this path depends on many things — e.g., the sequence of paths C_ϕ we choose, the gauge in which we are working —, but it depends on them continuously and therefore the homotopy class associated with the path in H_d is invariant. Thus the homotopy group $\pi_1(H_d)$ defines a topological charge of the string that is different from its $\mathbb{Z}_2$ flux. Since $\pi_1(H_d) = \mathbb{Z}$ this topological charge takes integer values.

Furthermore, the family $\{C_\phi C_{\phi=0}^{-1}\}$ is equivalent to the family of paths $\{C'_\phi\}$ shown in figure 7.3, that sweeps over the surface of a sphere. The sequence $\{C'_\phi\}$ traces out a path in H, or H_c if we assume no path C'_ϕ crosses a string loop or monopole. Lubkin (1963) showed that the associated element of $\pi_1(H)$ represents the value of the topological magnetic charge inside the two-sphere[1] Since there is a natural mapping from a path in $H_d(x_0)$ to a path in $H_c(x_0)$,

$$h(C_\phi, x_0) \to h(C_\phi, x_0)\; h^{-1}(C_{\phi=0}, x_0), \qquad 0 \leq \phi < 2\pi, \tag{7.1.2}$$

we can conclude that the element of $\pi_1(H_d) \cong \pi_1(H_c)$ is the same as the magnetic charge on the string loop.

More generally, in the presence of string loops and monopoles, we can define the magnetic charge inside any region R whose boundary ∂R is homeomorphic to S^2. The surface ∂R is associated with a closed path in H_c and so with a homotopy element of $\pi_1[H_c]$ that represents the magnetic charge within R. Continuously deforming the surface ∂R will change the path in H_d but not its associated

[1] A clear exposition of Lubkin's construction can be found in Coleman (1983).

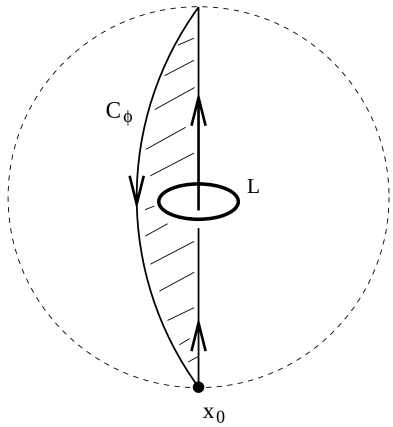


Figure 7.2: The family of closed paths $\{C_\phi \,|\, 0 \leq \phi < 2\pi\}$ sweeps out a degenerate torus that encloses the string loop L.

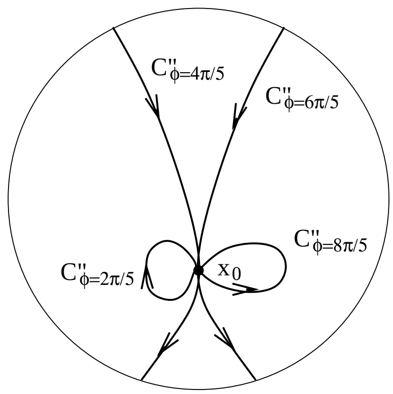


Figure 7.3: A family of loops C'_ϕ that sweeps over the surface of a sphere. The loops C'_0 and $C'_{2\pi}$ are degenerate.

homotopy element. Thus there is a homomorphism

$$h^{(1)} : \pi_2[\mathcal{M}, x_0] \to \pi_1[H_c(x_0)], \tag{7.1.3}$$

where $\mathcal{M}$ denotes the manifold that is obtained when all string loops and monopoles are removed from $\mathbb{R}^3$.

In the presence of Alice loops, it is important to specify a base point when defining the magnetic charge within a surface. A surface can be based in topologically inequivalent ways and the sign of the enclosed magnetic charge is dependent on this. The following will make this clear.

Consider the situation sketched in figure 7.4a, where we have an Alice loop L and a free surface $\bar{\Sigma}$ that is homeomorphic to S^2. The surface can be attached to a basepoint x_0 in many topologically inequivalent ways, two of which are illustrated in figure 7.4b and 7.4c.

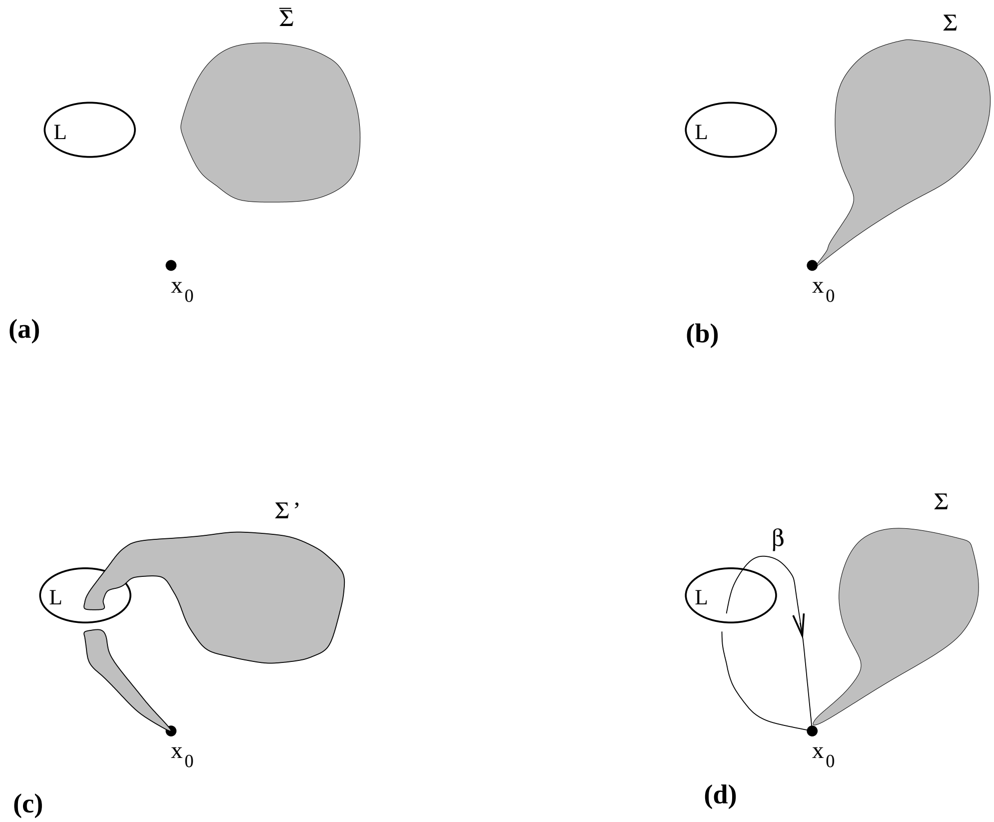


Figure 7.4: The free surface $\bar{\Sigma}$ in (a) can be threaded to the basepoint x_0 in inequivalent ways, two of which are illustrated in (b) and (c). The surface (c) can be deformed to (d), which differs from (b) by the degenerate tube β that begins and ends at the base point and goes once through the string loop L.

We can construct a correspondence between the two surfaces Σ and Σ' by means of a closed path β (figure 7.4d). The surface Σ' can be deformed into a "degenerate tube" β, joined to the surface Σ. Thus the surfaces Σ and Σ' "differ" by the closed path β, or more generally, by an homotopy element of $\pi_1[\mathcal{M}, x_0]$. Let us make this statement a bit more precise.

A surface based at x_0 can be defined by a mapping $\Sigma(u, v)$ of the unit square into $\mathcal{M}$,

$$\Sigma(u, v) \to \mathcal{M} \qquad 0 \le u, v \le 1, \tag{7.1.4}$$

subject to the restriction that Σ takes the entire circumference of the square into the basepoint x_0,

$$\Sigma(0, v) = \Sigma(1, v) = \Sigma(u, 0) = \Sigma(u, 1) = x_0. \tag{7.1.5}$$

Thus the surface is represented by closing up the border of the square. A closed path based at x_0 can be defined as a mapping $\beta(t)$ of the unit interval into $\mathcal{M}$,

$$\beta(t) \to \mathcal{M} \qquad 0 \le t \le 1, \tag{7.1.6}$$

with the constraint

$$\beta(0) = \beta(1) = x_0. \tag{7.1.7}$$

To define a mapping which takes Σ to a new surface Σ' by adding the path β to Σ

$$\tau_\beta : \Sigma \to \Sigma', \tag{7.1.8}$$

we can make the following construction (see figure 7.5). We take the the unit square in the $u-v$ plane and inscribe in it a square half as big at its center. On the inner square we define $\tau_\beta(\Sigma)$ to act precisely as Σ acts on its entire square, so that $\tau_\beta(\Sigma)$ takes the inner square into exactly the same surface as Σ takes its entire square into. We divide the remaining part of the square between the inner and outer circumferences into a sequence of square circumferences growing continuously from the outer to the inner one, as t goes from 0 to 1. We let $\tau_\beta(\Sigma)$ take the circumference parametrized by t into the single point $\beta(t)$.

Clearly, if Σ_1 and Σ_2 are in the same homotopy class of $\pi_2(\mathcal{M}, x_0)$, then $\tau_\beta(\Sigma_1)$ and $\tau_\beta(\Sigma_2)$ will be in the same homotopy class of $\pi_2(\mathcal{M}, x_0)$. In addition, if β_1 and β_2 are homotopic, then $\tau_{\beta_1}(\Sigma_1)$ and $\tau_{\beta_2}(\Sigma)$ will be in the same homotopy class of $\pi_2(\mathcal{M}, x_0)$. Thus the mapping τ defines a homomorphism

$$\tau : \pi_1[\mathcal{M}, x_0] \to \mathrm{Aut}(\pi_2[\mathcal{M}, x_0]), \tag{7.1.9}$$

that takes homotopy classes of closed paths to mappings of $\pi_2[\mathcal{M}, x_0]$ onto itself, so called automorphisms of $\pi_2[\mathcal{M}, x_0]$.

Changing the threading of a free surface to the basepoint modifies the magnetic charge enclosed by the surface. Recall that we defined the magnetic charge within a surface by a family of closed paths $\{C_\phi\}$ that sweeps over the surface. After adding a closed path β to a surface, the sequence of closed paths $\{C'_\phi\}$ defining the new surface can be related to the first sequence by

$$C'_\phi = \beta \circ C_\phi \circ \beta^{-1}, \tag{7.1.10}$$

i.e., a path C'_ϕ first completes the path β, then C_ϕ, and finally returns to its starting point via β^{-1}.

A closed path in $\mathcal{M}$ that starts and ends at x_0 is associated with an element of the group $H(x_0)$ by the equation (7.1.1). If we smoothly deform the path, the group element smoothly varies, but it will remain in the same connected component of the group. This implies that equation (7.1.1) defines a homomorphism

$$h^{(2)} : \pi_1[\mathcal{M}, x_0] \to \pi_0[H(x_0)]. \tag{7.1.11}$$

The paths C_ϕ and C'_ϕ can each be mapped to an element of $H_c(x_0)$. From (7.1.10) it follows that

$$h^{(2)}(C'_\phi) = h^{(2)}(\beta) h^{(2)}(C_\phi) h^{(2)}(\beta)^{-1}. \tag{7.1.12}$$

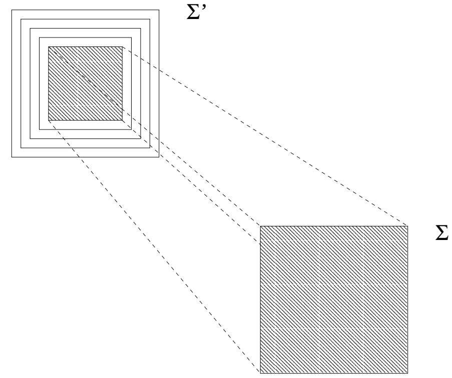


Figure 7.5: The shaded square in the lower right represents a surface Σ in $\mathcal{M}$ based at x_0. The entire circumference of the square is taken into the basepoint x_0. The square in the upper left represents the surface $\tau_\beta(\Sigma)$. The inner part of that square is just the shaded square, scaled down in size. It is surrounded by a sequence of square circumferences which expand outward to fill the rest of the square. Each circumference is taken into a single point along the path β, the innermost going into the ending-point and the outermost into the starting-point of β. Since the path β starts and ends at x_0, the new surface $\tau_\beta(\Sigma)$ is again based at x_0.

The sequences $\{C_\phi\}$ and $\{C'_\phi\}$ are each associated with a closed path in H_c defining the magnetic charges enclosed by the surfaces Σ and Σ', respectively (7.1.3). Equation (7.1.12) shows that the charges are related according to

$$h^{(1)}(\Sigma') = h^{(2)}(\beta)h^{(1)}(\Sigma)h^{(2)}(\beta)^{-1}. \tag{7.1.13}$$

Thus $h^{(1)}(\Sigma')$ is the closed path in H_c that is obtained when $h^{(2)}(\beta)$ acts on the closed path $h^{(1)}(\Sigma)$ by conjugation.

Consider again figure 7.4. The path β once encircles a string loop and so $h^{(2)}(\beta) \in H_d$. Since $h^{(1)}(\Sigma)$ is a path in H_c, equation (7.1.13) effectively says that the magnetic charges enclosed by Σ and Σ' differ by sign.

The preceding observations show that the sign of the magnetic charge enclosed within a surface is ambiguous, depending on how we have chosen to connect the surface to the basepoint. Still, we can keep track of the transfer of magnetic charge as long as we keep the basepoint fixed. In the next section we study a simple process of charge transfer.

7.2 Magnetic charge transfer

We study a simple process of magnetic charge transfer. We take an Alice loop and a magnetic monopole, and we carry the monopole through the loop. By this process, the magnetic charge of the monopole changes sign and magnetic charge is transferred to the loop.

The initial situation is depicted in figure 7.6a. The surface a_1 encloses the loop L and the surface a_2 encloses the monopole M. Both surfaces are based at

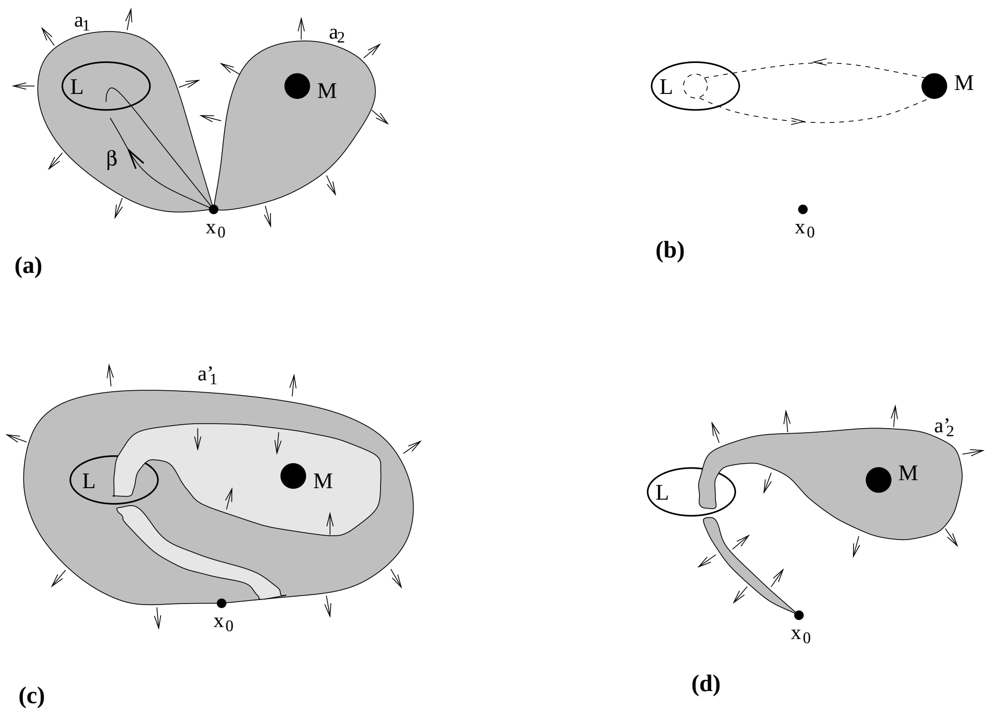


Figure 7.6: The magnetic flux of the string loops L_1 and L_2 is defined in terms of the paths β_1 and β_2 shown in (a), and the magnetic charges of the loops are defined in terms of the surfaces a_1 and a_2; the paths and surfaces are based at the point x_0. When L_2 goes through L_1 as in (b), the surface a_1' shown in (c) is dragged to a_1 and the surface a_2' shown in (d) is dragged to a_2. The arrows on the surfaces indicate outward-pointing normals.

x_0. The path β begins and ends at x_0 and encircles the loop. At present, the magnetic charge of the loop is given by $h^{(1)}(a_1)$ and the magnetic charge of the monopole is given by $h^{(1)}(a_2)$ (using the homomorphism (7.1.3)).

We carry the monopole M through the loop L along the path shown in 7.6b. What are the magnetic charges of the loop and the monopole at the end of this process? To determine this, consider the surfaces a_1' and a_2' in 7.6c-d. (The surface a_1' is composed of an outer surface enclosing the loop and monopole, and and an inner surface equal to $a_2'^{-1}$.) During the process, these surfaces are dragged back to a_1 and a_2 respectively, if the surfaces are deformed so that no surface ever touches the string loop or monopole. Therefore, the magnetic charge enclosed by a_1 after the process is the same as magnetic charge enclosed by a_1' before the process. Similarly, the magnetic charge enclosed by a_2 after the process is the same as the magnetic charge enclosed by a_2' before the process. So, to determine the magnetic charges of the loop and monopole after the process, we only need to determine the magnetic charges enclosed by a_1' and a_2' before the process.

First consider the surface a_2'. Deforming a_2' to the degenerate tube β plus the surface a_2, as depicted in figure 7.7a, shows that a_2' can be expressed as

$$a_2' = \tau_\beta(a_2) \tag{7.2.1}$$

where t_β is the automorphism of $\pi_2[\mathcal{M}, x_0]$, (7.1.8).

Figure 7.7b shows that surface a_1' can be expressed as the sum of two surfaces. The first surface, the outer surface, is $a_1 \circ a_2$, the surface that encloses both the loop and the monopole. The second surface, the inner surface, is $(a_2')^{-1}$, the same as a_2' except with the opposite orientation. So we can write a_1' as

$$a_1' = a_1 \circ a_2 \circ (a_2')^{-1}. \tag{7.2.2}$$

We can apply (7.1.13) to find the magnetic charges enclosed by a_1' and a_2'. For the latter we obtain

$$h^{(1)}(a_2') = h^{(2)}(\beta_1)h^{(1)}(a_2)h^{(2)}(\beta_1)^{-1}, \tag{7.2.3}$$

and for a_1'

$$h^{(1)}(a_1') = h^{(1)}(a_1 \circ a_2)h^{(1)}({a_2'}^{-1}). \tag{7.2.4}$$

Combining these results, we find the magnetic charges enclosed by a_2 and a_1 *after* the transfer process

$$h^{(1)'}(a_2) \equiv h^{(1)}(a_2') = -h^{(1)}(a_2), \tag{7.2.5}$$

since $h^{(2)}(\beta_1) \in H_d$, and

$$h^{(1)'}(a_1) \equiv h^{(1)}(a_1') = h^{(1)}(a_1)h^{(1)}(a_2)h^{(1)}(a_2), \tag{7.2.6}$$

since $h^{(1)}({a_2'}^{-1}) = \left[h^{(1)}(a_2')\right]^{-1} = -h^{(1)}(a_2') = h^{(1)}(a_2)$.

If the magnetic charge on the string loop is initially m_1 and the charge of the monopole m_2, then the equations (7.2.5) and (7.2.6) tell us that after the monopole has gone through the loop, the magnetic charge of the loop has become $m_1 + 2m_2$, and the charge of the monopole has become $-m_2$. As expected, the overall magnetic charge is conserved.

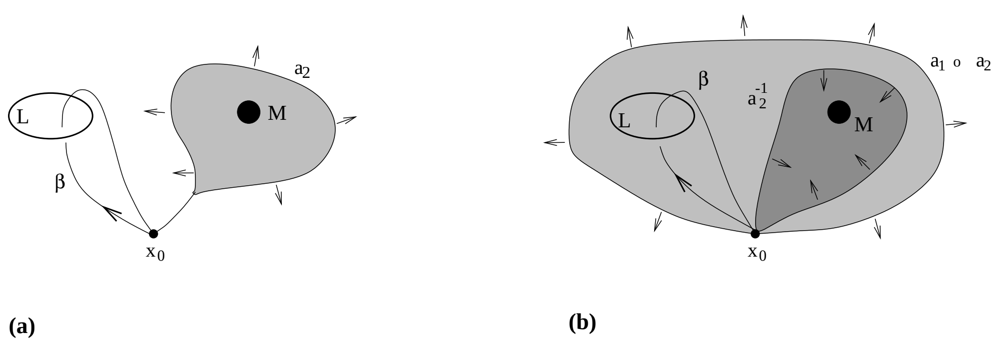


Figure 7.7: Deformations of the surfaces shown in figure 7.4c-d. In (a) the surface a_2' has been deformed to the degenerate tube β plus the surface a_2. In (b) the surface a_1' has been deformed to the surface $a_1 \circ a_2$ that encloses both loops, plus the inverse of a_2' (i.e., a_2' with the orientation reversed); the surface $(a_2')^{-1}$ is the sum of the degenerate tube $(\beta)^{-1}$ and the surface $(a_2)^{-1}$.

7.3 An ansatz for a magnetically charged string

We now turn to the construction of solutions for a magnetically charged Alice string. We only treat the first step: the generation of an ansatz.

The following considerations can help us find an ansatz. First, recall the ansatz for a "bare" string derived in chapter 4,

$$\Phi = m(r)\Phi_1 + q(r)\mathrm{e}^{T_1\theta/2}\Phi_2\mathrm{e}^{-T_1\theta/2}, \tag{4.2.3a}$$

$$A_\theta = \frac{\alpha(r)}{2er}T_1. \tag{4.2.3b}$$

This string carries a $\mathbb{Z}_2$ flux

$$h = P\exp\left(\oint_C A_\theta(r=\infty)\, r d\theta\right) = \exp\left(T_1\pi\right), \tag{7.3.1}$$

which is an element of H_d. The path C is a circle "at infinity" in the xy-plane for a particular value of z. For the "bare" string, it does not matter which z we take; the flux h is independent of z.

In the first section of this chapter we showed that a string is magnetically charged if its flux h traces out a closed path in H_d. Hence, to turn the "bare" string into a magnetically charged string, we must modify the ansatz (4.2.3) such that h becomes a periodic function of z.

The space H_d consists of rotations by 180° about axes in the T_1T_2-plane (see chapter 3). We can parametrise it as

$$H_d = \left\{\exp[(\mathrm{e}^{T_3 z}T_1\mathrm{e}^{-T_3 z})\pi] \,|\, 0 \le z \le 2\pi\right\} \tag{7.3.2}$$

since the conjugation of T_1 by $\mathrm{e}^{T_3 z}$ rotates T_1 about the T_3 axis by a phase z. Hence, to obtain a string with a flux that twists along the z axis, we can take the bare string ansatz and substitute T_1 with a generator $R_1(z)$,

$$R_1(z) = \mathrm{e}^{T_3\zeta(z)}T_1\mathrm{e}^{-T_3\zeta(z)}, \tag{7.3.3}$$

which is T_1 rotated about T_3 by a phase $\zeta(z)$. (Note that $\exp\left(\mathrm{e}^{T_3\zeta(z)}T_1\mathrm{e}^{-T_3\zeta(z)}\right) = \mathrm{e}^{T_3\zeta(z)}\mathrm{e}^{T_1}\mathrm{e}^{-T_3\zeta(z)}$.) This yields the ansatz

$$\Phi = \mathrm{e}^{R_1(z)\theta/2}\left[m(r,z)\Phi_1 + q(r,z)\Phi_2\right]\mathrm{e}^{-R_1(z)\theta/2}, \tag{7.3.4}$$

$$A_\theta = \frac{\alpha(r,z)}{2er}R_1(z). \tag{7.3.5}$$

Note that by simply replacing the z dependence for a t dependence, we obtain an ansatz for a string that carries a magnetic current.

Let us consider the boundary conditions that go with this ansatz. For $r \to \infty$, the boundary values for α, q and m are unchanged,

$$\alpha \to 1, \qquad q \to a, \qquad m \to a. \tag{7.3.6}$$

For $r \to 0$ we must have

$$\alpha \to 0, \qquad q \to 0, \qquad m \to 0. \tag{7.3.7}$$

In constrast with the solutions for a pure and an electrically charged string, the function m must go to zero for a magnetically charged string since it is combined with terms that depend on θ (7.3.4). So for this ansatz the order parameter vanishes at the core of the string and the $SO(3)$ symmetry is fully restored there. This property is the hallmark of a defect carrying magnetic charge.

The ansatz features a gauge field

$$F_{z\theta} = \frac{1}{2er}\frac{\partial \alpha}{\partial z} R_1(z) - \frac{\alpha}{2er}\frac{d\zeta}{dz} \mathrm{e}^{T_3 \zeta} T_2 \mathrm{e}^{-T_3 \zeta}. \tag{7.3.8}$$

Since the charge generator is

$$Q = \mathrm{e}^{R_1(z)\theta/2} T_3 \mathrm{e}^{-R_1(z)\theta/2}, \tag{7.3.9}$$

we find that the longe range gauge-invariant field is

$$\mathcal{F}_{z\theta} = \frac{1}{2er}\frac{d\zeta}{dz} \sin(\theta/2) \tag{7.3.10}$$

which however is double-valued. The double-valuedness leads one to suspect that the ansatz is not correct. This suspicion is confirmed if we substitute the ansatz in the energy functional and derive differential equations for α, q, m and ζ by extremizing the energy. The equations we obtain are long and complicated and more seriously, contain numerous unpleasant terms that are dependent on θ.

At this moment, we don't know how to correct the ansatz. Nevertheless, the above observations show that this is an interesting topic for further research.

Chapter 8

Conclusion

In the preceding chapters we have given a detailed account of Alice electrodynamics. We focussed on a simple Alice model with a $SO(3)$ symmetry spontaneously broken to $U(1) \rtimes \mathbb{Z}_2$. We discussed the properties of the topological defects of this model, Alice strings and magnetic monopoles.

We saw that loops of Alice string can carry charge. The standard interpretation of a charged Alice loop is that the loop carries unlocalized Cheshire charge: it has a long range electric (or magnetic) field from which its charge can be inferred, but the charge cannot be localized, neither on the string core nor in its vicinity.

We have presented a new view on charged Alice strings. We propose that the electric-magnetic dual symmetry is a priori not broken by the Alice string, but may be broken by either an electric or magnetic charge excitation of the string core. We suggest that there are two alternative ways to achieve this, which are each others dual image. On the one hand, an Alice string can carry electric charge, either as Cheshire charge produced by a magnetic supercurrent on the string, or as electric charge localized at the string core. On the other hand, an Alice string can carry magnetic charge, either as Cheshire charge produced by an electric supercurrent, or as magnetic charge localized at the string core.

We have constructed explicit solutions for Alice strings that support this view. We first presented a solution for a "pure" Alice string. We found that the Higgs field is combined of *two* functions, not one. In addition, it turned out that the Higgs field is non-zero at the origin, leading to an unbroken $U(1) \rtimes \mathbb{Z}_2$ symmetry at the origin, but the unbroken generator here is T_1 instead of T_3 at spatial infinity. This fact allows the Alice string to be turned into an electric superconductor.

We have constructed explicit solutions for an Alice string with localized electric charge and current. The existence of this solution shows that in spite of what has been commonly thought, Alice strings can carry electric charge localized at its core. The electric current produces a tangential magnetic field around the loop. However, this configuration does not correspond to a magnetic cheshire charge.

Finally, we have constructed an ansatz for an Alice string with magnetic charge and current. Although basically it has the correct form, it does not correspond with a consistent solution.

These results are important. They show that the current view on Alice electrodynamics is incomplete and they strongly support the new view we propose. But the results are not conclusive and further research is needed to complete the picture.

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