# Vector-valued modular forms in String Theory

JHW Seoul, 2008

P.B. and T. Gannon

# Introduction

Modular forms: holomorphic functions $f\colon \mathbf{H} \to \mathbb{C}$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2w} f(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$ and some (even) positive integer $w$.

$f(\tau)$ is periodic $\Rightarrow$ $q$-expansion (with $q = e^{2\pi i \tau}$)

$$f = \sum_{n \in \mathbb{Z}} f_n q^n$$

Classically: holomorphicity at $\tau = i\infty$.

More generally: finite order pole.

Example: Eisenstein series (for $k > 1$)

$$E_{2k}(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

where

$$\sigma_k(n) = \sum_{d|n} d^k$$

and $B_k$ is the $k$-th Bernoulli number.

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k}\, E_{2k}(\tau)$$

Any holomorphic modular form is a polynomial in $E_4$ and $E_6$.

Examples: $E_8 = E_4^2$ and $E_{10} = E_4 E_6$.

Discriminant form (weight 12)

$$\Delta = \frac{1}{1728}\left(E_4^3 - E_6^2\right) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

Doesn't vanish on **H**!

Hauptmodul (weight 0)

$$J(q) = \frac{E_4^3}{\Delta} - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots$$

Univalent map $\mathbf{H} \to \mathbb{CP}^1$, invariant under $SL_2(\mathbb{Z})$.

Any modular form of weight 0 is a polynomial in $J$.

Applications:

- function theory
- algebraic geometry (STWW)
- number theory (CFT)
- combinatorics (partitions)
- topology (elliptic cohomology)

- algebra (VOA-s, affine Lie, Moonshine)

- $\cdots$

- String Theory and (2d) CFT

Need for a theory of vector-valued modular forms.

# Vector-valued modular forms

Holomorphic maps $\mathbb{X}:\mathbf{H}\to\mathbb{C}^d$ such that

$$\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{2w}\,\rho\begin{pmatrix}a\,b\\c\,d\end{pmatrix}\mathbb{X}(\tau)$$

$\rho$: suitable (projective) representation of $\mathrm{SL}_2(\mathbb{Z})$ (more precisely, an ordinary representation of $B_3$).

$$\exp(-2\pi i\boldsymbol{\Lambda}\tau)\,\mathbb{X}(\tau)=\sum_{n\in\mathbb{Z}}\mathbb{X}[n]\,q^n\ ,$$

where $\exp(2\pi i\boldsymbol{\Lambda})=\rho\begin{pmatrix}1\,1\\0\,1\end{pmatrix}$ (exponent matrix).

Singular part: $\mathcal{P}\mathbb{X}=\sum_{n<0}\mathbb{X}[n]\,q^n$ .

$\mathbb{X}$ meromorphic at $\tau=i\infty$ if $\mathcal{P}\mathbb{X}\in\mathbb{C}\left[q^{-1}\right]^d$.

Thanks to $\Delta$, general case may be reduced to $w=0$, in which case $\rho$ is a true representation of $\mathrm{PSL}_2(\mathbb{Z})$.

$\mathcal{M}(\rho)$: linear space of weight 0 forms.

Singular part map $\mathcal{P}:\mathcal{M}(\rho)\rightarrow\mathbb{C}[q^{-1}]^d$ is affected by choice of $\mathbf{\Lambda}$.

Is there a "best" choice?

$\mathcal{P}$ bijective if and only if

$$\mathrm{Tr}(\mathbf{\Lambda}) = d - \frac{\alpha}{2} - \frac{\beta_1 + 2\beta_2}{3},$$

where $d, \alpha, \beta_1, \beta_2$ are nonnegative integers (signature of $\rho$).

# The fundamental matrix

Multiplication by $J(q)$ takes $\mathcal{M}(\rho)$ to itself $\Rightarrow$

$\mathcal{M}(\rho)$ is a $\mathbb{C}[J]$-module (of finite rank).

Fundamental matrix $\Xi(q)$:

1. Columns generate $\mathcal{M}(\rho)$;

2. $\Xi(q) \to q^{\mathbf{\Lambda}-1}$ as $q \to 0$.

Determinantal formula

$$\det \boldsymbol{\Xi}(q) = \left(\frac{E_4(q)}{\Delta(q)^{1/3}}\right)^{\beta_1+2\beta_2} \left(\frac{E_6(q)}{\Delta(q)^{1/2}}\right)^{\alpha}$$

Consequence: $\boldsymbol{\Xi}(q)$ is invertible.

$\boldsymbol{\Xi}(q)^{-1}\,\mathbb{X}(q) \in \mathbb{C}[J]^d$ for any $\mathbb{X}(q) \in \mathcal{M}(\rho)$.

Inversion formula

$$\mathcal{P}^{-1} = \frac{1}{2\pi i} \oint \frac{J'(z)}{J(q) - J(z)} \boldsymbol{\Xi}(q)\, \boldsymbol{\Xi}(z)^{-1}\, z^{\boldsymbol{\Lambda}} dz$$

Should we know $\boldsymbol{\Xi}(q)$, we would know everything about $\mathcal{M}(\rho)$.

Can we determine $\boldsymbol{\Xi}(q)$?

$\mathcal{M}(\rho)$ is a differential module:

$$\nabla = \frac{E_{10}(q)}{\Delta(q)}\, q\, \frac{\mathrm{d}}{\mathrm{d}q}$$

takes $\mathcal{M}(\rho)$ to itself $\Rightarrow$ the matrix entries of $\Xi(q)^{-1}\,\nabla\Xi(q)$ are (first order) polynomials in $J(q)$.

Differential equation:

$$\nabla\Xi(q) = \Xi(q)\,\mathcal{D}(q)\ ,$$

with

$$\mathcal{D}(q) = J(q)\,(\mathbf{\Lambda}-1) + \mathbf{\Lambda}\mathcal{X} - (\mathcal{X}+240)\,(\mathbf{\Lambda}-1)$$

and

$$\mathcal{X} = \lim_{q\to 0}\left(q^{-\mathbf{\Lambda}}\Xi(q) - q^{-1}\right)$$

(characteristic matrix).

Differential equation may be solved (recursively)

$\Rightarrow\mathcal{M}(\rho)$ charaterized by the pair $(\mathbf{\Lambda},\mathcal{X})$!

How can we determine $\mathcal{X}$?

Rademacher-Petersson formula: (poorly) convergent infinite sum.

Alternative approach: spectral condition.

New (uniformizing) variable $\mathfrak{z}(q)=\frac{984-J(q)}{1728}$.

Hypergeometric equation

$$\frac{\mathrm{d}\mathbf{\Xi}(\mathfrak{z})}{\mathrm{d}\mathfrak{z}}=\mathbf{\Xi}(\mathfrak{z})\left(\frac{\mathcal{A}}{2\mathfrak{z}}+\frac{\mathcal{B}}{3(\mathfrak{z}-1)}\right)$$

$$\begin{aligned}
\mathcal{A} &= \frac{31}{36}(1-\mathbf{\Lambda}) - \frac{1}{864}(\mathcal{X}+\mathbf{\Lambda}\mathcal{X}-\mathcal{X}\mathbf{\Lambda}) \\
\mathcal{B} &= \frac{41}{24}(1-\mathbf{\Lambda}) + \frac{1}{576}(\mathcal{X}+\mathbf{\Lambda}\mathcal{X}-\mathcal{X}\mathbf{\Lambda})
\end{aligned}$$

Analytic properties of $\mathbf{\Xi}(q) \Rightarrow$ spectral condition

$$\mathcal{A}(\mathcal{A}-1) = \mathcal{B}(\mathcal{B}-1)(\mathcal{B}-2) = 0$$

System of quadratic equations for $\mathcal{X}$ (given $\mathbf{\Lambda}$).

Overdetermined for $d < 6$.

# Examples:

1. Lee-Yang model ($c=-\frac{22}{5}$)

$$\mathbf{\Lambda}=\begin{pmatrix}\frac{59}{60} & \\ & \frac{11}{60}\end{pmatrix} \qquad S=\frac{1}{2\sin\left(\frac{\pi}{5}\right)}\begin{pmatrix}1 & 2\cos\left(\frac{2\pi}{5}\right)\\ 2\cos\left(\frac{2\pi}{5}\right) & -1\end{pmatrix}$$

$$q^{-\mathbf{\Lambda}}\mathbf{\Xi}(q)=q^{-1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}1 & 26999\\ 1 & -245\end{pmatrix}+q\begin{pmatrix}1 & 1820504\\ 0 & -113239\end{pmatrix}+$$

$$+q^2\begin{pmatrix}1 & 50387239\\ 1 & -6029989\end{pmatrix}+q^3\begin{pmatrix}2 & 859014394\\ 1 & -148368374\end{pmatrix}+\cdots$$

2. Ising model ($c = \frac{1}{2}$)

$$\mathbf{\Lambda} = \begin{pmatrix} \frac{47}{48} & & \\ & \frac{23}{48} & \\ & & \frac{1}{24} \end{pmatrix} \qquad S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$q^{-\mathbf{\Lambda}}\mathbf{\Xi}(q) = q^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2325 & 94208 \\ 1 & 275 & -4096 \\ 1 & -25 & -23 \end{pmatrix} +$$

$$+ q\begin{pmatrix} 1 & 60630 & 9515008 \\ 1 & 13250 & -1130496 \\ 1 & -4121 & 253 \end{pmatrix} + q^2\begin{pmatrix} 1 & 811950 & 356765696 \\ 1 & 235500 & -63401984 \\ 1 & -102425 & -1794 \end{pmatrix} + \cdots$$

# Outlook and open questions

- Correspondence $\mathcal{X} \leftrightarrow \rho$
- Integrality properties
- Infinite dimensional representations
- Automorphic forms for other groups
- Hecke operators
- Jacobi forms