Simple current symmetries in Rational Conformal Field Theories

RCFT	MTC
primary field	simple object
conformal weight	twist
fusion ring	Grothendieck ring
charge conjugate	dual object
simple current	invertible object

Simple currents

Schellekens and Yankielowicz; Intriligator; Fuchs, Runkel and Schweigert

1. Modular invariants (Kreuzer and Schellekens)

2. GSO projections (Fuchs, Schweigert and Walcher)

3. Simple current extensions (Fuchs, Schellekens and Schweigert)

4. Kernel of the modular representation

Modular representation: unitary matrices S and T such that

1. $S^t = S$ and $S^2 = C$

2. $T_p^q = \omega(p)\delta_p^q$

3. Modular relation $STS = T^{-1}ST^{-1}$

4. Verlinde's formula

$$N_{pqr} = \sum_s \frac{S_{ps}S_{qs}S_{rs}}{S_{0s}}$$

Simple currents form an abelian group G (''center'', ''Picard group'')

Fusion matrix is a permutation matrix!

$$[\Pi(\alpha)]_p^q = \delta_p^{\alpha q}$$

Verlinde: $S^{-1}\Pi(\alpha) S$ is diagonal

One-to-one correspondence with simple currents.

''Schur's lemma''

Question: which groups arise?

Answer depends on the category.

In Ab: $|G| < \infty$.

In AbPerm: $\exists$ regular orbit.

But $G < S_5$ generated by $(1,2)\,(3,4)$ and $(1,4)\,(2,3)$!

Need extra structure.

Monomial matrices

$$Y(\alpha,\beta) = \Pi(\alpha)\, S \Pi(\beta)\, S^{-1}$$

Matrix elements

$$[Y(\alpha,\beta)]_p^q = \vartheta(\beta)\,\frac{\omega(q)}{\omega(\beta q)}\delta_p^{\alpha q}$$

where

$$\begin{aligned} \vartheta : G &\rightarrow \mathbb{C}^* \\ \alpha &\mapsto \exp\left(2\pi\imath\Delta_{\alpha 0}\right) \end{aligned}$$

From $Y(1,\alpha\beta) = Y(1,\alpha)\, Y(1,\beta)$ follows

$$\frac{\omega(\alpha p)\,\omega(\beta p)}{\omega(p)\,\omega(\alpha\beta p)} = \frac{\vartheta(\alpha)\,\vartheta(\beta)}{\vartheta(\alpha\beta)}$$

Projective representation of $G \times G$

$$Y(\alpha_1, \beta_1)\, Y(\alpha_2, \beta_2) = \frac{\vartheta(\alpha_2)\, \vartheta(\beta_1)}{\vartheta(\alpha_2 \beta_1)} Y(\alpha_1 \alpha_2, \beta_1 \beta_2)$$

Cocycle condition

$$\vartheta(\alpha\beta)\, \vartheta(\beta\gamma)\, \vartheta(\gamma\alpha) = \vartheta(\alpha)\, \vartheta(\beta)\, \vartheta(\gamma)\, \vartheta(\alpha\beta\gamma)$$

ϑ is a quadratic function on G!

In unitary theories

$$\vartheta(\alpha^n) = \vartheta(\alpha)^{n^2}$$

for $n \in \mathbb{Z}$, i.e. ϑ is a quadratic form.

Modular representation: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto M$

Commutation rule

$$M^{-1} Y(\alpha,\beta) M = \frac{\vartheta(\alpha)^{b(c-a)}\,\vartheta(\beta)^{c(b-d)}}{\vartheta(\alpha\beta)^{bc}} Y\left(\alpha^a\beta^c, \alpha^b\beta^d\right)$$

Modular invariants:

$$Z(H,\varepsilon) = \frac{1}{|H|} \sum_{\alpha,\beta\in H} \varepsilon(\alpha,\beta)\, Y(\alpha,\beta)$$

where $H < G$, and $\varepsilon : H \times H \rightarrow \mathbb{C}^*$ is a bihomomorphism such that $\varepsilon(\alpha,\alpha) = \vartheta(\alpha)^{-1}$ (''discrete torsion'').

1. Exponent of G divides the order of T.

2. Symmetry relation

$$Y\left(\alpha,\beta\right)^t = SY\left(\beta,\alpha\right)S^{-1}$$

3. "Curious formula"

$$\operatorname{Tr} Y\left(\alpha,\beta\right) = \sum_p \phi_p\left(\alpha,\beta\right)$$

where

$$\left|\phi_p\left(\alpha,\beta\right)\right| = \begin{cases} 1 & \text{if both } \alpha \text{ and } \beta \text{ fix } p, \\ 0 & \text{otherwise.} \end{cases}$$

Weighted permutation actions

(G, ϑ): quadratic group

$$\vartheta(\alpha\beta)\,\vartheta(\beta\gamma)\,\vartheta(\gamma\alpha) = \vartheta(\alpha)\,\vartheta(\beta)\,\vartheta(\gamma)\,\vartheta(\alpha\beta\gamma)$$

(X, ω): weighted permutation action.

Support X: finite G-set.

Weight function $\omega : X \to \mathbb{C}^*$

$$\frac{\omega(\alpha p)\,\omega(\beta p)}{\omega(p)\,\omega(\alpha\beta p)} = \frac{\vartheta(\alpha)\,\vartheta(\beta)}{\vartheta(\alpha\beta)}$$

Example: regular WPA with $X = G$ and $\omega = \vartheta$.

Morphisms, sums, transitivity.

Decomposition into transitives.

Radical: $\text{rad}(G, \vartheta) = \left(\sqrt{G}, \sqrt{\vartheta}\right)$

$\sqrt{G} = \{\alpha \in G \,|\, \vartheta\,(\alpha\beta) = \vartheta\,(\alpha)\,\vartheta\,(\beta) \text{ for all } \beta \in G\}$

$\sqrt{\vartheta}$ = restriction of ϑ to $\sqrt{G}$ (character of $\sqrt{G}$)

$$\begin{cases} \text{nondegenerate} & \text{if } \left|\sqrt{G}\right| = 1 \\ \text{completely degenerate} & \text{if } \sqrt{G} = G \end{cases}$$

Classification of transitives: coset WPA-s $\mathcal{W}[H,\xi]$

$\xi \in \mathrm{Irr}\left(\sqrt{G}\right)$ and $H < \ker\left(\sqrt{\vartheta}\xi\right)$.

Regular WPA $= \mathcal{W}[1,\xi_0]$.

WPA-s of $(G,\vartheta) \rightleftharpoons$ WPA-s of $\mathrm{rad}(G,\vartheta)$!

$\mathrm{mult}_{\mathcal{X}}(H,\xi)$ = multiplicity of $\mathcal{W}[H,\xi]$ in $\mathcal{X}$.

$$\Upsilon(\alpha,\beta) = \vartheta(\beta) \sum_{p \in \mathrm{Fix}(\alpha)} \frac{\omega(p)}{\omega(\beta p)}$$

For the coset WPA $\mathcal{W}[H,\xi]$

$$\Upsilon(\alpha,\beta)=\begin{cases}\xi(\beta)\,[G:H] & \text{if } \alpha\in H \text{ and } \beta\in\sqrt{G},\\ 0 & \text{otherwise.}\end{cases}$$

Associated monomial matrices

$$[Y(\alpha,\beta)]_p^q=\vartheta(\beta)\,\frac{\omega(q)}{\omega(\beta q)}\delta_p^{\alpha q}$$

Projective representation of $G\times G$

$$Y(\alpha_1,\beta_1)\,Y(\alpha_2,\beta_2)=\frac{\vartheta(\alpha_2)\,\vartheta(\beta_1)}{\vartheta(\alpha_2\beta_1)}Y(\alpha_1\alpha_2,\beta_1\beta_2)$$

Trace formula

$$\mathrm{Tr}\,Y(\alpha,\beta)=\Upsilon(\alpha,\beta)$$

Admissibility

1. Galois invariance:

$$\mathsf{mult}_{\mathcal{X}}\left(H,\xi^{l}\right)=\mathsf{mult}_{\mathcal{X}}\left(H,\xi\right)$$

   for any l coprime to the exponent of G.

2. Reciprocity:

$$\Upsilon\left(\alpha,\beta\right)=\Upsilon\left(\beta,\alpha\right)$$

3. Fixed point bound:

$$\left|\Upsilon\left(\alpha,\beta\right)\right|\leq\left|\mathsf{Fix}\left(\alpha\right)\cap\mathsf{Fix}\left(\beta\right)\right|$$

1. Galois invariance:

   (a) Exponent of G divides N (= the order of T).

   (b) Galois action => matrix $M = S^{-1}T^{l}ST^{m}ST^{l}$ is monomial if $lm \equiv 1 \pmod{N}$.

   (c) $M^{-1}Y(1,\beta)M = Y\left(1,\beta^{l}\right)$

2. Reciprocity: $Y(\alpha,\beta)^{t} = SY(\beta,\alpha)S^{-1}$

3. Fixed point bound: $\mathrm{Tr}\,Y(\alpha,\beta) = \sum_{p}\phi_{p}(\alpha,\beta)$

Simple current WPA: admissible that contains the regular!

Independent constraints.

Reciprocity => $\Upsilon(\alpha,\beta)$ integral.

Linear homogeneous conditions on multiplicities!

Admissible WPA-s: finitely generated semigroup (Minkowski-Weyl), but not freely generated.

Irreducible WPA = minimal admissible.

Decomposition into irreducibles is not unique.

Examples

1. $\mathcal{G} = (\mathbb{Z}_2, \xi_0)$
   3 transitives: F, R and W

   $$n_F F \oplus n_R R \oplus n_W W$$

   admissible if $n_R = n_W$
   2 irreducibles: F and $R \oplus W$.

2. $\mathcal{G} = (\mathbb{Z}_2, \xi_1)$
   3 transitives: F, R and W
   Admissibility condition: $n_R = n_F + n_W$
   2 irreducibles: $W \oplus R$ and $F \oplus R$ (Ising).

3. $\mathcal{G} = (\mathbb{Z}_2 \times \mathbb{Z}_2, \xi_0)$
   11 transitives
   6 irreducibles: 1, 4, 4, 4, 11*, 16*

4. $\mathcal{G} = (\mathbb{Z}_3 \times \mathbb{Z}_3, \xi_0)$
   22 transitives
   14 Galois orbits
   8 irreducibles: 1, 9, 9, 9, 9, 35*, 69*, 81*
   1 relation ($35 \oplus 35 = 1 \oplus 69$)

5. $\mathcal{G} = (\mathbb{Z}_2 \times \mathbb{Z}_4, \xi_0)$
   27 transitives
   23 Galois orbits
   23 irreducibles
   12 relations

Ocneanu: ”Main problem is not the language but

(i) the construction of interesting examples;

(ii) understanding of the underlying structures;

(iii) classification results!”

1. Simple current groups

2. Modular invariants

$$Z(H,\varepsilon) = \frac{1}{|H|} \sum_{\alpha,\beta\in H} \varepsilon(\alpha,\beta)\, Y(\alpha,\beta)$$

ε: cohomological data from quadratic group

$Y(\alpha,\beta)$: combinatorial data from WPA

Can enumerate all simple current invariants for (G,ϑ)!

$(\mathbb{Z}_2, \xi_0)$	modular invariants
F	(1)
$R \oplus W$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

$(\mathbb{Z}_2, \xi_1)$	modular invariants
$R \oplus F$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$
$R \oplus W$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$

3. Modular data

WPA restricts T, S and the fusion rules.

$N_{pq}^{r} > 0$ implies $\vartheta(\alpha)\,\omega(p)\,\omega(q)\,\omega(\alpha r) = \omega(\alpha p)\,\omega(\alpha q)\,\omega(r)$

$N_{\alpha p} = N_p Y\left(\alpha^{-1}, 1\right) = Y\left(\alpha^{-1}, 1\right) N_p$

Most effective for irreducible WPA-s.

E.g. for $R \oplus F$ of $(\mathbb{Z}_2, \xi_1)$

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$