Vector-valued modular functions and forms

Let $\mathbf{\Gamma} = \mathrm{SL}_2(\mathbb{Z})$ be the classical modular group of 2×2 integer matrices with unit determinant, and let $\rho : \mathbf{\Gamma} \to \mathrm{GL}_d(\mathbb{C})$ denote a d-dimensional matrix representation of $\mathbf{\Gamma}$ satisfying some technical assumptions: namely, that $T := \rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonal and $\rho\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the identity matrix. It follows from the above properties that the matrices $S := \rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U := \rho\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = ST^{-1}$ have orders 2 and 3 respectively, consequently their characteristic polynomials read $z^{d-\alpha}(z-1)^{\alpha}$ and $z^{d-\beta_1-\beta_2}(z-\omega)^{\beta_1}(z-\overline{\omega})^{\beta_2}$ for suitable integers α, β_1 and β_2, where $\omega = \exp\left(\frac{2\pi i}{3}\right)$. The 4-tuple $(d, \alpha, \beta_1, \beta_2)$ is a most important characteristic of the representation ρ, called its *signature*. Note that the signature may be computed from the relations

$$\mathrm{Tr}(S) = d - 2\alpha \,, \tag{1a}$$

$$\mathrm{Tr}(U) = d - \frac{3}{2}(\beta_1 + \beta_2) + i\frac{\sqrt{3}}{2}(\beta_1 - \beta_2) \,. \tag{1b}$$

The group $\mathbf{\Gamma}$ acts on the complex upper half-plane $\mathbf{H} = \{\tau \in \mathbb{C} \,|\, \mathsf{Im}\,\tau > 0\}$ by fractional linear transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \,, \tag{2}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{\Gamma}$. The quotient of the upper half-plane by this group action becomes after compactification (i.e. after adding the cusp at $\tau = i\infty$) a Riemann-surface of genus 0. By a (weak) *vector-valued modular form* of *weight* w and *multiplier* ρ is meant a map $\mathbb{X} : \mathbf{H} \to \mathbb{C}^d$ that satisfies the following criteria.

1.Holomorphicity: $\mathbb{X}$ is holomorphic in the upper half-plane $\mathbf{H}$;

2.Functional equation: for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{\Gamma}$

$$\boxed{\mathbb{X}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{w}\,\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbb{X}(\tau)} \tag{3}$$

3.Cuspidal behavior: $\mathbb{X}$ has finite order poles at the cusp $\tau = i\infty$.

To explain this last point, note that for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathbf{\Gamma}$ the functional equation Eq.3 takes the form

$$\mathbb{X}(\tau + 1) = T\mathbb{X}(\tau) \,. \tag{4}$$

Since T is diagonal by assumption, there exists a diagonal matrix $\mathbf{\Lambda}$ (called the *exponent matrix*) such that $T = \exp(2\pi i \mathbf{\Lambda})$. It follows that

$\exp(-2\pi i\tau\mathbf{\Lambda})\,\mathbb{X}(\tau)$ is periodic (with period 1), consequently it may be expanded into a Fourier-series, which reads

$$q^{-\mathbf{\Lambda}}\mathbb{X}(q)=\sum_{n\in\mathbb{Z}}\mathbb{X}[n]\,q^n\ , \tag{5}$$

using the usual notation $q=\exp(2\pi i\tau)$. The *principal* (or *singular*) *part* $\mathcal{P}\mathbb{X}$ of $\mathbb{X}$ is the sum of the terms with negative powers of q on the right-hand side of Eq.5, i.e.

$$\mathcal{P}\mathbb{X}:=\sum_{n<0}\mathbb{X}[n]\,q^n\ . \tag{6}$$

With this definition, $\mathbb{X}$ has finite order poles at the cusp $\tau=i\infty$ if and only if its principal part $\mathcal{P}\mathbb{X}$ is a finite sum.

In case ρ is the trivial one-dimensional representation, we recover the standard notion of (weak) modular forms. In case the weight w is zero, the corresponding forms are called *vector-valued modular functions*: these form an infinite dimensional linear space $\mathcal{M}(\rho)$ over $\mathbb{C}$. Taking principal parts provides us with a linear map $\mathcal{P}:\mathcal{M}(\rho)\rightarrow q^{-1}\mathbb{C}[q^{-1}]^d$, which may be shown to be bijective under suitable conditions.
At this point, we should notice that the exponent matrix $\mathbf{\Lambda}$ is not completely determined by the representation ρ. Indeed, only the fractional part of its diagonal elements are determined, but the integer parts might be chosen at will. But this choice affects the definition Eq.6 of the principal part! It may be shown that $\mathcal{P}$ is bijective provided one chooses the integer part of $\mathbf{\Lambda}$ in such a way that the *trace condition*

$$\boxed{\mathrm{Tr}(\mathbf{\Lambda})=d-\frac{\alpha}{2}-\frac{\beta_1+2\beta_2}{3}} \tag{7}$$

is satisfied, and this may always be achieved.

From now on, we shall suppose that an exponent matrix satisfying Eq.7 has been chosen, consequently $\mathcal{P}$ is bijective. This means that we can choose a *canonical basis* $\mathbb{X}^{(\xi;n)}$ of $\mathcal{M}(\rho)$ such that

$$\left[\mathcal{P}\mathbb{X}^{(\xi;n)}\right]_\eta=q^{-n}\delta_{\xi\eta}\ , \tag{8}$$

i.e. $\mathbb{X}^{(\xi;n)}$ has a pole of order $n>0$ in the ξ-th component. The basic problem is to determine the q dependence of the $\mathbb{X}^{(\xi;n)}$.

Let

$$J(q)=q^{-1}+\sum_{n=1}^{\infty}c(n)\,q^n=q^{-1}+196884q+\dots \tag{9}$$

denote the Hauptmodul of $\mathbf{\Gamma}$, i.e. the (suitably normalized) modular invariant. Multiplication by J takes the space $\mathcal{M}(\rho)$ to itself, in other words $\mathcal{M}(\rho)$ is a $\mathbb{C}[J]$-module. The important point is that this is a $\mathbb{C}[J]$-module of finite rank, because the canonical basis vectors satisfy the *recursion relations*

$$\mathbb{X}^{(\xi;m+1)} = J(q)\,\mathbb{X}^{(\xi;m)} - \sum_{n=1}^{m-1} c(n)\,\mathbb{X}^{(\xi;m-n)} - \sum_{\eta} \mathcal{X}_{\eta}^{(\xi;m)}\mathbb{X}^{(\eta;1)}\ , \tag{10}$$

where

$$\mathcal{X}_{\eta}^{(\xi;m)} := \lim_{q\to 0}\left(\left[q^{-\mathbf{\Lambda}}\mathbb{X}^{(\xi;m)}\right]_{\eta} - q^{-m}\delta_{\xi\eta}\right) \tag{11}$$

denotes the "constant part" of $\mathbb{X}^{(\xi;m)}$. These recursion relations allow to express each canonical basis vector $\mathbb{X}^{(\xi;m)}$ in terms of the $\mathbb{X}^{(\xi;1)}$-s, proving that the latter generate the $\mathbb{C}[J]$-module $\mathcal{M}(\rho)$.

Let's introduce the *fundamental matrix*

$$\mathbf{\Xi}(q)_{\xi\eta} := \left[\mathbb{X}^{(\eta;1)}\right]_{\xi}\ , \tag{12}$$

whose columns generate the $\mathbb{C}[J]$-module $\mathcal{M}(\rho)$. It may be shown that the determinant of this matrix is given by

$$\det\mathbf{\Xi}(\tau) = \left(\frac{E_4(\tau)}{\Delta(\tau)^{1/3}}\right)^{\beta_1+2\beta_2}\left(\frac{E_6(\tau)}{\Delta(\tau)^{1/2}}\right)^{\alpha}\ , \tag{13}$$

where

$$E_4 = 1 + 240q + 2160q^2 + \ldots\ , \tag{14a}$$

$$E_6 = 1 - 504q - 16632q^2 - \ldots \tag{14b}$$

denote the Eisenstein series of weights 4 and 6, and $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is the discriminant form of weight 12. In particular, the fundamental matrix is invertible except at the elliptic points $\tau = i$ and $\tau = \omega$. In terms of the fundamental matrix, any $\mathbb{X} \in \mathcal{M}(\rho)$ may be written as $\mathbb{X}(q) = \mathbf{\Xi}(q)\,\mathbf{X}(q)$, where $\mathbf{X} \in \mathbb{C}[J]^{\,d}$ is a column vector whose components are polynomials in the Hauptmodul $J(q)$. Even better, one has the following

Inversion formula: *if $\mathbb{X}(q) \in \mathcal{M}(\rho)$ has principal part $\mathcal{P}\mathbb{X}$, then*

$$\mathbb{X}(q) = \mathbf{\Xi}(q)\,\frac{1}{2\pi i}\oint \frac{J'(z)}{J(q)-J(z)}\mathbf{\Xi}(z)^{-1}\,z^{\mathbf{\Lambda}}\mathcal{P}\mathbb{X}(z)\,dz\ . \tag{15}$$

Here $J'(z) = -z^{-2} + \sum_{n=1}^{\infty} nc(n)\,z^{n-1}$ is the derivative of the Hauptmodul J, and the integral is to be taken over a closed contour encircling the origin, and contained in the circle of radius $|q|$.

In particular,

$$\left[\mathbb{X}^{(\xi;n)}(q)\right]_{\eta} = \frac{1}{2\pi i}\oint \frac{z^{\mathbf{\Lambda}_{\xi\xi}-n}J'(z)}{J(q)-J(z)}\left[\mathbf{\Xi}(q)\,\mathbf{\Xi}(z)^{-1}\right]_{\eta\xi}dz\ . \tag{16}$$

As can be seen from the above, the whole story boils down to the computation of the fundamental matrix $\Xi(q)$. In this respect, the important observation is that the differential operator

$$\nabla := \frac{\mathcal{E}(\tau)}{2\pi i}\frac{\mathsf{d}}{\mathsf{d}\tau} \tag{17}$$

maps $\mathcal{M}(\rho)$ to itself, where

$$\mathcal{E}(q) := \frac{E_4(q)\,E_6(q)}{\Delta(q)} = \sum_{n=-1}^{\infty} \mathcal{E}_n q^n \ . \tag{18}$$

Looking at the action of ∇ on the canonical basis vectors, one gets the *differential relations*

$$\nabla \mathbb{X}^{(\xi;m)} = (\mathbf{\Lambda}_{\xi\xi} - m) \sum_{n=-1}^{m-1} \mathcal{E}_n \mathbb{X}^{(\xi;m-n)} + \sum_{\eta} \mathbf{\Lambda}_{\eta\eta} \mathcal{X}_{\eta}^{(\xi;m)} \mathbb{X}^{(\eta;1)} \ . \tag{19}$$

But the fundamental matrix $\Xi(q)$ determines completely the canonical basis vectors $\mathbb{X}^{(\xi;m)}$ through Eq.16: inserting these expressions into Eq.19, one arrives at the result that the differential relations are compatible with the solution Eq.16 of the recursion relations if, and only if the fundamental matrix satisfies the first order linear differential equation – the *compatibility equation* –

$$\boxed{\frac{1}{2\pi i}\frac{\mathsf{d}\Xi(\tau)}{\mathsf{d}\tau} = \Xi(\tau)\,\mathfrak{D}(\tau)}\,, \tag{20}$$

where

$$\mathfrak{D}(\tau) = \frac{1}{\mathcal{E}(\tau)} \left\{ (J(\tau) - 240)\,(\mathbf{\Lambda} - 1) + \mathcal{X} + \mathbf{\Lambda}\mathcal{X} - \mathcal{X}\mathbf{\Lambda} \right\} \tag{21}$$

and $\mathcal{X}_{\xi\eta} = \mathcal{X}_{\xi}^{(\eta;1)}$ is the so-called *characteristic matrix*. Taking into account the boundary condition

$$\Xi(q) \to q^{\mathbf{\Lambda}-1} \quad \text{as} \quad q \to 0\,, \tag{22}$$

which follows from Eq.8 , one can solve Eq.20 provided one knows the exponent matrix $\mathbf{\Lambda}$ and the characteristic matrix $\mathcal{X}$. By the theory of ordinary differential equations, Eq.20 has series solutions that converge in suitably small neighborhoods; since $\Xi(\tau)$ solves the equation and is holomorphic on $\mathbf{H}$, it follows that the above series converge everywhere.

Note that Eq.20 has singular points at the poles of $\mathfrak{D}(\tau)$, i.e. at the cusp $\tau = i\infty$ and at the elliptic points $\tau = i$ and $\tau = \exp\left(\frac{2\pi i}{3}\right)$. As it turns out, all these are regular singular points. To see this, let's consider the uniformizing function

$$\mathfrak{z}(\tau) = \frac{984 - J(\tau)}{1728}\ , \tag{23}$$

which maps the upper half-plane $\mathbf{H}$ onto the complex plane $\mathbb{C}$, has valence 2 and 3 at the elliptic points, and has a first order pole at the cusp $\tau = i\infty$; as usual, we extend the definition of $\mathfrak{z}$ so that it maps $\tau = i\infty$ to ∞. As a function of $\mathfrak{z}$, the fundamental matrix satisfies the following form of the compatibility equation

$$\boxed{\frac{\mathrm{d}\Xi(\mathfrak{z})}{\mathrm{d}\mathfrak{z}} = \Xi(\mathfrak{z})\left(\frac{\mathcal{A}}{2\mathfrak{z}} + \frac{\mathcal{B}}{3\left(\mathfrak{z}-1\right)}\right)}\,, \tag{24}$$

with

$$\mathcal{A} = \frac{31}{36}\left(1-\mathbf{\Lambda}\right) - \frac{1}{864}\left(\mathcal{X} + \mathbf{\Lambda}\mathcal{X} - \mathcal{X}\mathbf{\Lambda}\right)\,, \tag{25a}$$

$$\mathcal{B} = \frac{41}{24}\left(1-\mathbf{\Lambda}\right) + \frac{1}{576}\left(\mathcal{X} + \mathbf{\Lambda}\mathcal{X} - \mathcal{X}\mathbf{\Lambda}\right)\,. \tag{25b}$$

The differential equation Eq.24 has indeed three regular singular points (at $\mathfrak{z} = 0$, 1 and ∞, corresponding to $\tau = i$, ω and $i\infty$ respectively), hence it is a (matrix-valued) hypergeometric equation.

As a function of $\mathfrak{z}$, the fundamental matrix is *not* single valued: its multivaluedness is described by the monodromy of Eq.24 , which is in turn determined by the representation ρ. The monodromy group of the hypergeometric equation Eq.24 is precisely the image of ρ, e.g. the monodromy around $\mathfrak{z} = 0$ is given by $S = \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, while that around $\mathfrak{z} = 1$ by $U = \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$.

The coefficient matrices $\mathcal{A}$ and $\mathcal{B}$ in Eq.24 are far from being arbitrary, they are severely restricted by the analytic properties of the fundamental matrix. In particular, they are both semisimple (i.e. diagonalizable), and satisfy the

Spectral condition: *the possible eigenvalues of $\mathcal{A}$ are 0 or 1, while those of $\mathcal{B}$ are either 0, 1 or 2.*

Note that the multiplicities of the different eigenvalues might be expressed in terms of the signature, which leads to

$$\det\left(z-\mathcal{A}\right) = z^{d-\alpha}\left(z-1\right)^{\alpha}\,, \tag{26a}$$

$$\det\left(z-\mathcal{B}\right) = z^{d-\beta_1-\beta_2}\left(z-1\right)^{\beta_1}\left(z-2\right)^{\beta_2}\,. \tag{26b}$$

This, in turn, gives

$$\mathrm{Tr}(\mathcal{A}) = \alpha\,, \tag{27a}$$

$$\mathrm{Tr}(\mathcal{B}) = \beta_1 + 2\beta_2\,, \tag{27b}$$

$$\mathrm{Tr}\left(\mathcal{B}^2\right) = \beta_1 + 4\beta_2\,, \tag{27c}$$

and the algebraic relations

$$\mathcal{A}\left(\mathcal{A}-1\right) = \mathcal{B}\left(\mathcal{B}-1\right)\left(\mathcal{B}-2\right) = 0\,. \tag{28}$$

Combining these with the relation

$$1 - \mathbf{\Lambda} = \frac{\mathcal{A}}{2} + \frac{\mathcal{B}}{3} \tag{29}$$

that follows from Eqs.25a and 25b , one concludes that [1]

$$\mathrm{Tr}(\mathcal{X}) = 4\,(62\beta_1 + 124\beta_2 - 123\alpha), \tag{30}$$

and that for a given diagonal exponent matrix $\mathbf{\Lambda}$ the matrix $\mathcal{A}$ has to satisfy

$$\begin{aligned} \mathcal{A}^2 &= \mathcal{A}\,, \\ \mathcal{A}\mathbf{\Lambda}\mathcal{A} &= -\frac{17}{18}\mathcal{A} - 2\left(\mathcal{A}\mathbf{\Lambda}^2 + \mathbf{\Lambda}\mathcal{A}\mathbf{\Lambda} + \mathbf{\Lambda}^2\mathcal{A}\right) \\ &\quad +3\,(\mathcal{A}\mathbf{\Lambda} + \mathbf{\Lambda}\mathcal{A}) - 4\mathbf{\Lambda}^3 + 8\mathbf{\Lambda}^2 - \frac{44}{9}\mathbf{\Lambda} + \frac{8}{9}\,, \end{aligned} \tag{31}$$

which is a system of quadratic equations for the matrix elements of $\mathcal{A}$. Once a solution to Eq.31 is known, the corresponding characteristic matrix may be determined from Eq.25a.

[1]Formally, one also gets the trace condition Eq.7 , but this was assumed to hold right from the start.