EXACT SEQUENCES AND THE COMBINATORICS OF CONFORMAL MODELS

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Abstract. We investigate the mutual relations between the centers of different elements in the deconstruction lattice of a 2D conformal model, and show that these are best described by suitable exact sequences of abelian groups. In particular, we exhibit a long exact sequence connecting the centers of higher central quotients and/or extensions, that proves helpful in actual computations.

1. Introduction

It has been understood for a long time that a substantial amount of information about a rational conformal model [1, 2] is encoded in its fusion algebra, which describes the possible couplings between primary fields, i.e. which primaries (together with their conformal descendants) could appear in the operator product of any two given primaries. A most famous result in this direction, instigating important mathematical developments over the years, is the celebrated formula of Verlinde [3] that relates the fusion rules of the model, i.e. the structure constants of the fusion algebra, to the matrix describing the transformation properties of the chiral characters under the modular transformation $S:\tau\rightarrow{}^{-1}/\tau$, and which allows to reconstruct the latter from the knowledge of the fusion rules and the conformal weights of the primaries. But it is fair to say that this is just the tip of the iceberg, for several similar relations are known, e.g. for the Frobenius-Schur indicators of the primaries [4], or for the traces of finite order mapping classes [5].

As has been discussed in [6, 7], an interesting structure related to the fusion algebra results from the consideration of collections of primaries that, besides containing the vacuum, are closed under the fusion product. These may be shown to form a lattice with many nice properties, like being self-dual and modular [8, 9]. Some elements of this so-called deconstruction lattice (the local ones) may be identified with the representation rings of suitable finite groups, and this allows transferring to them several standard notions from group theory, like commutativity, nilpotency, etc., and these turn out to make perfectly good sense for generic elements.

In particular, there is a way to define the notions of center and of central quotients and extensions [6], which correspond to the standard group theoretic notions in case of local elements. The importance of all this stems from the fact that there is a well understood relationship between many properties of central quotients and extensions, and this can facilitate greatly the analysis of specific models. A notable exception to this is the structure of the center itself, as there is no obvious connection between the center of central quotients and/or extensions, in complete analogy with the case of groups. The aim of the present note is to show that a useful characterization of this relationship can be given in terms of exact sequences of abelian groups [10].

In order to be accessible to a wider readership, we briefly summarize background material on the deconstruction lattice in Section 2, on central quotients and extensions in Section 3, and on exact sequences in Section 4. Then we turn to our main subject, and study in Section 5 the restriction homomorphism that connects the centers of different elements in the deconstruction lattice. The most important results can be found in Section 6, which investigates the Galois correspondence between subgroups of the center and central quotients, resulting in exact sequences able to describe the subtleties of this relationship. The theme of Section 7 is a long exact sequence connecting the centers of higher quotients that proves useful in actual computations, while Section 8 is concerned with the case of central extensions instead of quotients, based on the dual nature of these two notions. In the last section, we summarize the results, and comment on open questions and possible future developments. Finally, as the modularity of the deconstruction lattice plays a pivotal role in some of the results of Section 6, a streamlined simple proof of this fundamental result is presented in the Appendix, based on an explicit characterization of the join operation that could prove interesting in itself.

2. The deconstruction lattice

Let's consider a 2D rational conformal model [1, 2] with a finite number of primaries. We shall denote by N_{pq}^{r} the fusion rule coefficients, that is the multiplicity of a primary r in the fusion product of the primaries p and q. The collection of all those subsets of primaries that contain the vacuum and are closed under the fusion product (meaning that if $N_{pq}^{r}>0$ with both p and q belonging to it, then r does also belongs to it) may be shown [6, 7] to form a modular lattice $\mathscr{L}$, termed the *deconstruction lattice* because of its fundamental role in the classification of the different orbifold deconstructions of the model [11, 12]. The ordering in $\mathscr{L}$ is simply set inclusion, and the meet operation is set intersection (the join operation being less obvious, but see the Appendix).

Recall [6] that to each $\mathfrak{g}\in\mathscr{L}$ one can assign a partition of the primaries of the model into so-called $\mathfrak{g}$-classes characterized by the fact that the irreducible representations of the fusion algebra corresponding to different elements in the same class coincide when restricted to the elements of $\mathfrak{g}$. Of utmost importance is the $\mathfrak{g}$-class that contains the vacuum, the *trivial class* $\mathfrak{g}^{\perp}$, which may be shown to be itself an element of $\mathscr{L}$. It is straightforward that $\mathfrak{g}^{\perp}\subseteq\mathfrak{h}^{\perp}$ whenever $\mathfrak{h}\subseteq\mathfrak{g}$, and that the trivial class of $\mathfrak{g}^{\perp}$ is $\mathfrak{g}$ itself, hence the lattice $\mathscr{L}$ is *self-dual*, i.e. endowed with an order-reversing and involutive *duality map* $\mathfrak{g}\mapsto\mathfrak{g}^{\perp}$ that relates the join and meet operations via $(\mathfrak{g}\vee\mathfrak{h})^{\perp}=\mathfrak{g}^{\perp}\cap\mathfrak{h}^{\perp}$.

As it turns out, self-dual lattices are closely related to undirected graphs (with possible loops): any such graph determines a self-dual lattice, and any self-dual lattice comes from a suitable graph[1]. A fairly non-intuitive result is that the deconstruction lattice of a conformal model corresponds to its locality graph, i.e. the graph whose vertices are the primary fields, with two of them adjacent if they are mutually local, i.e. if their OPE is single-valued. Actually, instead of the locality graph one can use the so-called locality diagram, whose vertices correspond to equilocality classes of primaries, i.e. collections of primaries that are mutually local

[1]Note that quite different graphs may lead to the same lattice, but the collection of all such graphs may be characterized in a simple manner [13].

with the same set of primaries, providing a nice graphical representation of the deconstruction lattice [14, 13].

Especially important are those elements $\mathfrak{g}\in\mathscr{L}$, termed *local* ones, for which $\mathfrak{g}\subseteq\mathfrak{g}^{\perp}$, since these provide the input data for the orbifold deconstruction procedure [11, 12], and correspond to the different orbifold realizations of the given model. In particular, for each local $\mathfrak{g}\in\mathscr{L}$ there exists a finite group, the twist group of the corresponding orbifold, whose representation ring (viewed as a λ-ring, i.e. taking into account the different possible symmetrizations of tensor powers) is isomorphic[2] with $\mathfrak{g}$. What is more, if $\mathfrak{g}\in\mathscr{L}$ is local and $\mathfrak{h}\in\mathscr{L}$ is contained in it, then $\mathfrak{h}$ is local too, and the group corresponding to $\mathfrak{h}$ is a homomorphic image (i.e. factor group) of the one corresponding to $\mathfrak{g}$. An important feature of local $\mathfrak{g}\in\mathscr{L}$ is that all of their elements, besides having (rational) integer quantum dimension, have either integer or half-integer conformal weight.

Let us note that in case of abelian models, when all the primaries are simple currents [16, 17, 18], i.e. primaries of quantum dimension 1, the fusion closed sets are nothing but the different subgroups of the group of simple currents, hence the deconstruction lattice is isomorphic to the subgroup lattice of the latter, with the duality map being determined by the relation of mutual locality (which is in turn determined by the distribution of conformal weights). In such a case a simple current extension by integer spin simple currents [19] has the same result as the corresponding orbifold deconstruction of the model[3].

It should transpire from the above that the deconstruction lattice $\mathscr{L}$ is an important algebraic/combinatorial invariant of the conformal model under study: in particular, it provides a concise description of the different realizations of the given model as an orbifold of another one. Understanding the structure of $\mathscr{L}$ through the relation of its elements to each other could provide information about the model that would be hard to obtain by other means. One such piece of information is related to the notion of central quotients and extensions, to which we now turn.

3. Central quotients and extensions

Given $\mathfrak{g}\in\mathscr{L}$, an important numerical characteristic of any $\mathfrak{g}$-class $\mathtt{C}$ is its *mass*

$$\mu(\mathtt{C})=\sum_{p\in\mathtt{C}}\mathtt{d}_p^2 \tag{3.1}$$

where $\mathtt{d}_p$ denotes the quantum dimension of the primary p. An especially important role is played by those $\mathfrak{g}$-classes whose mass is minimal (equal to that of the trivial class $\mathfrak{g}^{\perp}$), the *central classes*, which may be shown to form an abelian group $\mathtt{Z}(\mathfrak{g})$, the *center* of $\mathfrak{g}$, that permutes the collection of all $\mathfrak{g}$-classes, and whose identity element is the trivial class. For local $\mathfrak{g}\in\mathscr{L}$ corresponding to the finite group G, the center $\mathtt{Z}(\mathfrak{g})$ is isomorphic with the center of G. To each central class $\mathtt{z}\in\mathtt{Z}(\mathfrak{g})$ one may associate a complex-valued function $\varpi_{\mathtt{z}}:\mathfrak{g}\to\mathbb{C}$, its *central characte*r, such that for $\alpha,\beta\in\mathfrak{g}$ one has $\varpi_{\mathtt{z}}(\gamma)=\varpi_{\mathtt{z}}(\alpha)\,\varpi_{\mathtt{z}}(\beta)$ whenever $N_{\alpha\beta}^{\gamma}>0$, and one may show [6] that the map that assigns to each central class its central character provides an explicit isomorphism between $\mathtt{Z}(\mathfrak{g})$ and its Pontryagin dual.

[2] Note that this does not fix the group uniquely, because of the existence of so-called Brauer-tuples, i.e. non-isomorphic groups with identical character tables and power maps [15]. But such exceptions are rather sparse, and the resulting ambiguity can be handled.

[3] More generally, simple current extensions of arbitrary conformal models can be understood as orbifold deconstructions for $\mathfrak{g}\in\mathscr{L}$ consisting purely of integer spin simple currents.

It turns out that each subgroup $H < \mathtt{Z}(\mathfrak{g})$ determines another element of $\mathscr{L}$, the *central quotient* $\mathfrak{g}/H$, and there is a one-to-one correspondence between elements $\mathfrak{h} \in \mathscr{L}$ that satisfy $\mathfrak{g}/H \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ and subgroups of H. In particular, each $\mathfrak{g} \in \mathscr{L}$ has a *maximal central quotient*

$$\boldsymbol{\partial}\mathfrak{g} = \mathfrak{g}/\,\mathtt{Z}(\mathfrak{g}) \tag{3.2}$$

that is contained in all central quotients, and each $\mathfrak{h} \in \mathscr{L}$ that satisfies $\boldsymbol{\partial}\mathfrak{g} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ is a central quotient of $\mathfrak{g}$, i.e. $\mathfrak{h} = \mathfrak{g}/H$ for some subgroup $H < \mathtt{Z}(\mathfrak{g})$. Actually, the relevant subgroup reads $H = \{\mathsf{z} \in \mathtt{Z}(\mathfrak{g}) \,|\, \mathsf{z} \subseteq \mathfrak{h}^{\perp}\}$, and this implies that the union of the central classes contained in H (viewed as collections of primaries) equals $\mathfrak{h}^{\perp}$, i.e. $\cup H = \mathfrak{h}^{\perp}$. In particular,

$$\cup \mathtt{Z}(\mathfrak{g}) = (\boldsymbol{\partial}\mathfrak{g})^{\perp} \tag{3.3}$$

The other way round, a *central extension* of $\mathfrak{g} \in \mathscr{L}$ is an element $\mathfrak{h} \in \mathscr{L}$ of which $\mathfrak{g}$ is a central quotient. Using the duality map $\mathfrak{g} \mapsto \mathfrak{g}^{\perp}$ of $\mathscr{L}$, one can show [6] that there is a one-to-one correspondence between central extensions of $\mathfrak{g}$ and subgroups of $\mathtt{Z}(\mathfrak{g}^{\perp})$. In particular, there is a *maximal central extension* $\mathfrak{g}^{\intercal} = (\boldsymbol{\partial}(\mathfrak{g}^{\perp}))^{\perp}$ that contains all central extensions of $\mathfrak{g}$, and each $\mathfrak{h} \in \mathscr{L}$ that satisfies $\mathfrak{g} \subseteq \mathfrak{h} \subseteq \mathfrak{g}^{\intercal}$ is a central extension of $\mathfrak{g}$, i.e. $\mathfrak{g}$ is a central quotient of $\mathfrak{h}$; put another way, the inclusion $\boldsymbol{\partial}\mathfrak{g} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ is equivalent to $\mathfrak{h} \subseteq \mathfrak{g} \subseteq \mathfrak{h}^{\intercal}$.

The map that assigns to an element $\mathfrak{g} \in \mathscr{L}$ its maximal central quotient (resp. extension) $\boldsymbol{\partial}\mathfrak{g}$ (resp. $\mathfrak{g}^{\intercal}$) is order-preserving, i.e. $\boldsymbol{\partial}\mathfrak{h} \subseteq \boldsymbol{\partial}\mathfrak{g}$ (resp. $\mathfrak{h}^{\intercal} \subseteq \mathfrak{g}^{\intercal}$) whenever $\mathfrak{h} \subseteq \mathfrak{g}$. What is more interesting is the fact that one has

$$\boldsymbol{\partial}(\mathfrak{g}^{\intercal}) \subseteq \mathfrak{g} \subseteq (\boldsymbol{\partial}\mathfrak{g})^{\intercal} \tag{3.4}$$

for every $\mathfrak{g} \in \mathscr{L}$. Indeed, since $\mathfrak{g}$ is a central quotient of $\mathfrak{g}^{\intercal}$, it should contain the maximal central quotient of the latter, and in the same vein, since $\boldsymbol{\partial}\mathfrak{g}$ is a central quotient of $\mathfrak{g}$, the latter is a central extension of $\boldsymbol{\partial}\mathfrak{g}$, consequently it has to be contained in $(\boldsymbol{\partial}\mathfrak{g})^{\intercal}$. Actually, one may show that the map $\mathfrak{g} \mapsto (\boldsymbol{\partial}\mathfrak{g})^{\intercal}$ is a closure operation on $\mathscr{L}$, whose fixed points are those elements of $\mathscr{L}$ that are themselves maximal central extensions, i.e. of the form $\mathfrak{g}^{\intercal}$ for some $\mathfrak{g} \in \mathscr{L}$.

The importance of central quotients and extensions stems from the fact that their properties are determined to a large extent by group theory. For example, the mass of the quotient $\mathfrak{g}/H$ is $|H|$ times that of $\mathfrak{g}$,

$$\boldsymbol{\mu}(\mathfrak{g}/H) = |H|\,\boldsymbol{\mu}(\mathfrak{g}) \tag{3.5}$$

and one has a fairly good description of the classes of the quotient $\mathfrak{g}/H$ in terms of the classes of $\mathfrak{g}$ and the action of $H < \mathtt{Z}(\mathfrak{g})$ on them [6]. Based on this, one expects that the center of $\mathfrak{g} \in \mathscr{L}$ and that of its quotients $\mathfrak{g}/H$ are related somehow. As we shall show in the sequel, such a relation does indeed exist, but to describe it neatly we shall need the machinery of exact sequences [10].

As an illustration of the above, consider the minimal $N=2$ superconformal model of central charge $c=2$. The Hasse diagram of its deconstruction lattice is depicted in Figure 1 on page 5, each element being labeled by (the isomorphism type of) its center. Inspecting the figure, one recognizes that it is made up of two sublattices related by the duality map (which in this case is a flip, i.e. a 180° rotation around the center of the diagram), each one isomorphic with the subgroup lattice of the abelian group $\mathbb{Z}_{12} \times \mathbb{Z}_2$, the latter being nothing but the group of simple currents[4].

[4]It is a general fact that the center of the maximal element of the deconstruction lattice is isomorphic with the group of simple currents.

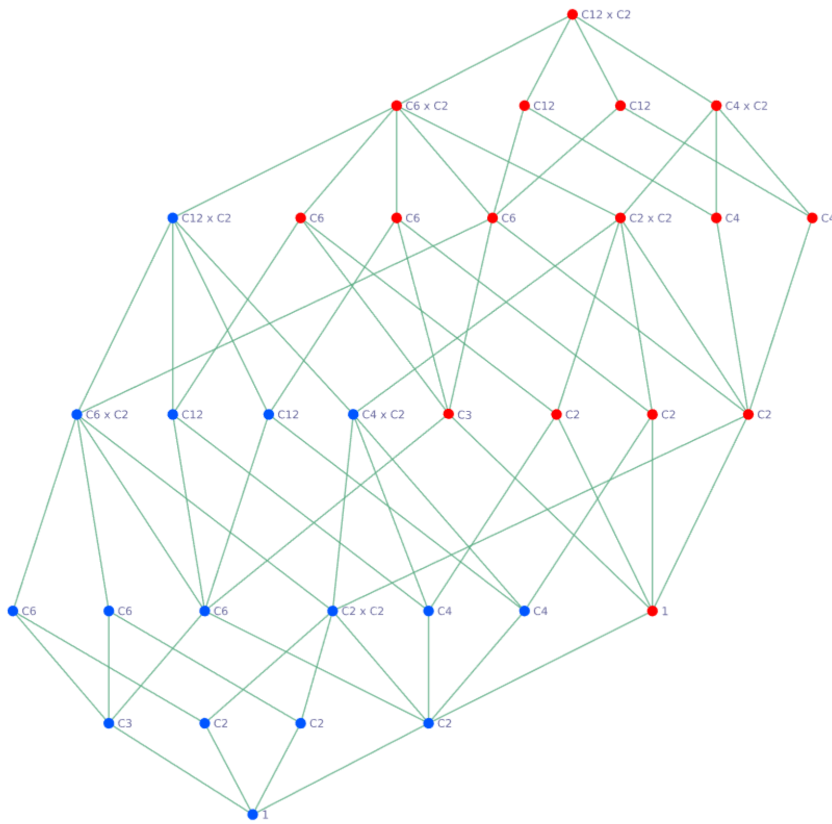

FIGURE 1. Deconstruction lattice of the minimal $N=2$ superconformal model of central charge $c=2$, the nodes being labeled by their centers, with $\mathtt{C}n$ denoting a cyclic group of order n.

But the picture is usually much more complicated, as exemplified by the $\mathbb{Z}_2$ orbifold of the compactified boson at radius $R=6$. The Hasse diagram of its deconstruction lattice is depicted in Figure 2 on page 6; once again, the duality map is the flip around the center of the diagram, and the nodes are labeled by their centers. While parts of the diagram mimic the subgroup lattice of some abelian group, the overall pattern is clearly more complicated than in the previous example. It was the desire to understand the structure underlying these patterns that motivated the present work.

Finally, let's remark that there is no reason to stop at the maximal central quotient $\boldsymbol{\partial}\mathfrak{g}$, for one can also consider the maximal central quotient of the latter, and so on repeatedly. Defining $\boldsymbol{\partial}^k\mathfrak{g}$ for $\mathfrak{g}\in\mathscr{L}$ and a positive integer k via $\boldsymbol{\partial}^1\mathfrak{g}=\boldsymbol{\partial}\mathfrak{g}$ and the recursion $\boldsymbol{\partial}^{k+1}\mathfrak{g}=\boldsymbol{\partial}(\boldsymbol{\partial}^k\mathfrak{g})$, we get a descending chain

$$\mathfrak{g}\supseteq\boldsymbol{\partial}^1\mathfrak{g}\supseteq\cdots\supseteq\boldsymbol{\partial}^n\mathfrak{g} \tag{3.6}$$

of elements of $\mathscr{L}$, an analogue of the upper central series from group theory [10]. In case the lattice $\mathscr{L}$ is finite, the inclusions in Eq.(3.6) imply that the sequence stabilizes after finitely many steps, i.e. there exists some integer N such that $\boldsymbol{\partial}^{N-1}\mathfrak{g}\neq\boldsymbol{\partial}^N\mathfrak{g}=\boldsymbol{\partial}^{N+1}\mathfrak{g}$. We shall denote by $\boldsymbol{\partial}^\infty\mathfrak{g}$ this last term of the upper central

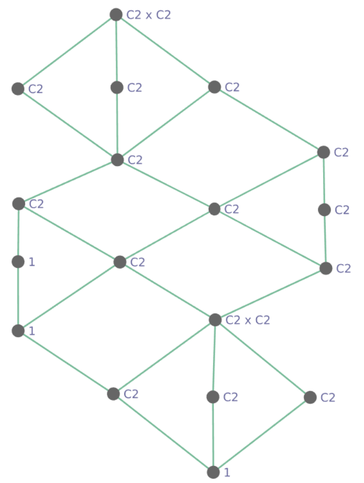

FIGURE 2. Deconstruction lattice of the $\mathbb{Z}_2$ orbifold of the compactified boson at radius $R=6$, the nodes being labeled by their centers.

series Eq.(3.6), and call it the *hypercentral quotient*[5] of $\mathfrak{g}$. Note that, because $\boldsymbol{\partial}^{\infty}\mathfrak{g}$ equals its own maximal central quotient by definition, it has trivial center, $\mathtt{Z}(\boldsymbol{\partial}^{\infty}\mathfrak{g})=1$, and this means that any element of $\mathscr{L}$ may be obtained by repeated central extensions from one with trivial center. As the structure of central extensions is well under control, this helps to reduce the study of subtler aspects of $\mathscr{L}$ to that of its centerless elements.

4. Exact sequences

As already alluded to previously, our goal is to understand the relations between the centers of different elements of the deconstruction lattice, in particular those that are central quotients or extensions of each other, and a most efficient way to express these connections is through the use of exact sequences. Since the latter are not part of the everyday toolkit of physicists, let's briefly summarize those basics facts about them that we shall need in what follows.

Recall [10] that an *exact sequence* of (abelian) groups is a sequence of group homomorphisms $\phi_i : A_{i-1} \to A_i$ for $i=1,\ldots,n$, such that the kernel of ϕ_{i+1} equals the image of ϕ_i for $1\leq i<n$; note that the domain of ϕ_{i+1} equals the range of ϕ_i. Their usual notation is

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} A_{n-1} \xrightarrow{\phi_n} A_n \tag{4.1}$$

In many cases one is not really interested in the actual homomorphisms that connect the A_i, but only in the relationship between them that follows from the existence of the exact sequence; in such cases one usually drops any explicit reference to the ϕ_i (unless the specification of some of them provides useful extra information).

[5]This terminology stems from the fact that for local $\mathfrak{g}\in\mathscr{L}$ that corresponds to the finite group G, the hypercentral quotient $\boldsymbol{\partial}^{\infty}\mathfrak{g}$ is itself local, and corresponds to the factor group of G by its hypercenter [10].

The most ubiquitous are the so-called *short exact sequences*

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1 \tag{4.2}$$

where 1 denotes the (isomorphism class of the) trivial group of order 1. The homomorphism theorem [20, 10] tells us at once that the existence of such an exact sequence is tantamount to the existence of a subgroup $K < B$, the kernel of the homomorphism from B to C, which satisfies $K \cong A$ and $C \cong B/K$.

Next in line come the four-term exact sequences

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 1 \tag{4.3}$$

which play an important role e.g. in algebraic number theory [21], and whose existence tells us that there exists a homomorphism $\phi : B \to C$ whose kernel is isomorphic with A, and whose cokernel $\mathtt{coker}(\phi) = C/\mathtt{im}(\phi)$ is isomorphic with D (note that, since every subgroup of an abelian group is normal, the cokernel makes perfectly good sense for homomorphism of abelian groups). In particular, it follows from Lagrange's theorem [10] that the orders of the groups appearing in a four-term exact sequence are related by $|A||C| = |B||D|$.

Finally, there are the l*ong exact sequences*

$$1 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \longrightarrow 1 \tag{4.4}$$

whose existence signals a more subtle and less immediate relation between the A_i. Note that long exact sequences are equivalent to a suitable collection of overlapping short exact sequences. To see how this comes about, let's consider the exact sequences

$$1 \longrightarrow A_1 \longrightarrow \cdots \xrightarrow{\phi_{n-1}} A_n \xrightarrow{\psi} B \longrightarrow 1 \tag{4.5}$$

and

$$1 \longrightarrow B \xrightarrow{\iota} A_{n+1} \xrightarrow{\phi_{n+1}} \cdots \longrightarrow A_{n+m} \longrightarrow 1 \tag{4.6}$$

with $B \neq 1$ (should B be trivial, the above sequences would shorten and would not have a common anchor). Then, since ψ is surjective with $\mathtt{ker}(\psi) = \mathtt{im}(\phi_{n-1})$, while ι is injective with $\mathtt{im}(\iota) = \mathtt{ker}(\phi_{n+1})$, the composite $\phi_n = \iota \circ \psi$ satisfies $\mathtt{ker}(\phi_n) = \mathtt{ker}(\psi) = \mathtt{im}(\phi_{n-1})$ and $\mathtt{im}(\phi_n) = \mathtt{im}(\iota) = \mathtt{ker}(\phi_{n+1})$, hence the sequence

$$1 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \xrightarrow{\phi_{n-1}} A_n \xrightarrow{\phi_n} A_{n+1} \xrightarrow{\phi_{n+1}} \cdots \longrightarrow A_{n+m} \longrightarrow 1$$

is exact. This trick allows us to combine exact sequences into longer ones, or to decompose a long exact sequence into a collection of short exact sequences. We shall encounter the application of these ideas in the later sections.

5. The restriction homomorphism

After all these preliminaries, it is time to focus on our basic problem, namely the relation between the center of an element $\mathfrak{g} \in \mathscr{L}$ and the centers of its different central quotients/extensions. As it turns out, it is worth to consider the more general problem of relating the center of $\mathfrak{g} \in \mathscr{L}$ to that of any $\mathfrak{h} \in \mathscr{L}$ contained in it. That this approach is meaningful is based on the following two results from [6]:

1. if $\mathfrak{h} \subseteq \mathfrak{g}$, then any $\mathfrak{h}$-class is a union of $\mathfrak{g}$-classes;
2. an $\mathfrak{h}$-class that contains a central $\mathfrak{g}$-class is itself central.

It follows that for any $\mathfrak{g},\mathfrak{h}\in\mathscr{L}$ with $\mathfrak{h}\subseteq\mathfrak{g}$ there is a unique central $\mathfrak{h}$-class that contains a given central $\mathfrak{g}$-class $\mathtt{z}\in\mathtt{Z}(\mathfrak{g})$.

Theorem 1. *For $\mathfrak{g},\mathfrak{h}\in\mathscr{L}$ with $\mathfrak{h}\subseteq\mathfrak{g}$, the map that assigns to each central $\mathfrak{g}$-class the unique $\mathfrak{h}$-class that contains it is a homomorphism $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}:\mathtt{Z}(\mathfrak{g})\rightarrow\mathtt{Z}(\mathfrak{h})$, with image $\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\left\{\mathtt{z}\in\mathtt{Z}(\mathfrak{h})\,|\,\mathtt{z}\cap(\boldsymbol{\partial}\mathfrak{g})^{\perp}\neq\emptyset\right\}$ and kernel $\mathtt{Z}(\mathfrak{g}|\mathfrak{h})=\{\mathtt{z}\in\mathtt{Z}(\mathfrak{g})\,|\,\mathtt{z}\subseteq\mathfrak{h}^{\perp}\}$.*

Proof. Let's denote by $\mathtt{z}^{\mathfrak{h}}$ the $\mathfrak{h}$-class containing the central $\mathfrak{g}$-class $\mathtt{z}\subseteq\mathtt{Z}(\mathfrak{g})$. Recall from Section 3 that to each central class one can assign its central character, and this provides an explicit isomorphism between the center and its Pontryagin dual. Because $\mathtt{z}\subseteq\mathtt{z}^{\mathfrak{h}}$ and $\mathfrak{h}\subseteq\mathfrak{g}$ implies $\boldsymbol{\varpi}_{\mathtt{z}^{\mathfrak{h}}}(\alpha)=\boldsymbol{\varpi}_{\mathtt{z}}(\alpha)$ for $\alpha\in\mathfrak{h}$, it follows that $\boldsymbol{\varpi}_{(\mathtt{z}_1\mathtt{z}_2)^{\mathfrak{h}}}(\alpha)=\boldsymbol{\varpi}_{\mathtt{z}_1\mathtt{z}_2}(\alpha)=\boldsymbol{\varpi}_{\mathtt{z}_1}(\alpha)\,\boldsymbol{\varpi}_{\mathtt{z}_2}(\alpha)=\boldsymbol{\varpi}_{\mathtt{z}_1^{\mathfrak{h}}}(\alpha)\,\boldsymbol{\varpi}_{\mathtt{z}_2^{\mathfrak{h}}}(\alpha)$ for $\mathtt{z}_1,\mathtt{z}_2\in\mathtt{Z}(\mathfrak{g})$. Since a central class is uniquely determined by its central character, this shows that $(\mathtt{z}_1\mathtt{z}_2)^{\mathfrak{h}}=\mathtt{z}_1^{\mathfrak{h}}\mathtt{z}_2^{\mathfrak{h}}$, proving that $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}$ is indeed a homomorphism. Now $\mathtt{z}\in\mathtt{ker}(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}})$, i.e. $\mathtt{z}^{\mathfrak{h}}=\mathfrak{h}^{\perp}$ precisely when $\mathtt{z}\subseteq\mathfrak{h}^{\perp}$, and this proves that $\mathtt{ker}(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}})=\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$. But a central $\mathfrak{h}$-class belongs to the image of $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}$ in case it contains at least one central $\mathfrak{g}$-class, i.e. if its intersection with $\cup\mathtt{Z}(\mathfrak{g})$ is not empty, one finally concludes that $\mathtt{im}(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}})=\left\{\mathtt{z}\in\mathtt{Z}(\mathfrak{h})\,|\,\mathtt{z}\cap(\boldsymbol{\partial}\mathfrak{g})^{\perp}\neq\emptyset\right\}=\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ by using Eq.(3.3). □

Notice that one has $\mathtt{Z}(\mathfrak{g}|\mathfrak{g})=1$ and $\mathtt{I}(\mathfrak{g}|\mathfrak{g})=\mathtt{Z}(\mathfrak{g})$ as a consequence of Eq.(3.3), while $\mathtt{Z}(\mathfrak{g}|\mathfrak{h})=\mathtt{Z}(\mathfrak{g})$ and $\mathtt{I}(\mathfrak{g}|\mathfrak{h})=1$ for $\mathfrak{h}\subseteq\boldsymbol{\partial}\mathfrak{g}$, and in particular $\mathtt{Z}(\mathfrak{g}|\boldsymbol{\partial}\mathfrak{g})=\mathtt{Z}(\mathfrak{g})$. Moreover, it follows from Theorem 1 and the homomorphism theorem [10] that

$$\mathtt{I}(\mathfrak{g}|\mathfrak{h})\cong\mathtt{Z}(\mathfrak{g})/\mathtt{Z}(\mathfrak{g}|\mathfrak{h}) \tag{5.1}$$

and by Lagrange's theorem this implies that the order of $\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ (being a homomorphic image, resp. subgroup) must divide both the order of $\mathtt{Z}(\mathfrak{g})$ and that of $\mathtt{Z}(\mathfrak{h})$. In particular, if the order of $\mathtt{Z}(\mathfrak{g})$ and of $\mathtt{Z}(\mathfrak{h})$ are coprime, then $\mathtt{I}(\mathfrak{g}|\mathfrak{h})=1$, hence $\mathtt{Z}(\mathfrak{g}|\mathfrak{h})=\mathtt{Z}(\mathfrak{g})$, and this implies $(\boldsymbol{\partial}\mathfrak{g})^{\perp}=\cup\mathtt{Z}(\mathfrak{g})\subseteq\mathfrak{h}^{\perp}$, i.e. $\mathfrak{h}\subseteq\boldsymbol{\partial}\mathfrak{g}$.

We shall call $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}:\mathtt{Z}(\mathfrak{g})\rightarrow\mathtt{Z}(\mathfrak{h})$ the *restriction homomorphism*. Clearly, for $\mathfrak{h}\subseteq\mathfrak{j}\subseteq\mathfrak{g}$ the restriction homomorphisms $\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}$ and $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}$ can be composed, and this observation leads to the following basic result.

Lemma 1. $\mathtt{Z}(\mathfrak{g}|\mathfrak{j})<\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ *and* $\mathtt{I}(\mathfrak{g}|\mathfrak{h})<\mathtt{I}(\mathfrak{j}|\mathfrak{h})$ *in case* $\mathfrak{h}\subseteq\mathfrak{j}\subseteq\mathfrak{g}$*, and the diagram*

$$\begin{array}{ccc} \mathtt{Z}(\mathfrak{g}) & \xrightarrow{\quad\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}\quad} & \mathtt{Z}(\mathfrak{h}) \\ & {\scriptstyle\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}}\searrow \quad \nearrow{\scriptstyle\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}} & \\ & \mathtt{Z}(\mathfrak{j}) & \end{array}$$

is commutative, hence $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}(\mathtt{I}(\mathfrak{g}|\mathfrak{j}))=\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ *and* $\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))=\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$.

Proof. $\mathfrak{h}\subseteq\mathfrak{j}\subseteq\mathfrak{g}$ implies $\mathfrak{j}^{\perp}\subseteq\mathfrak{h}^{\perp}$ and $(\boldsymbol{\partial}\mathfrak{g})^{\perp}\subseteq(\boldsymbol{\partial}\mathfrak{j})^{\perp}$, hence $\mathtt{z}\in\mathtt{Z}(\mathfrak{g}|\mathfrak{j})$ implies $\mathtt{z}\in\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$, while $\mathtt{z}\in\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ gives $\mathtt{z}\in\mathtt{I}(\mathfrak{j}|\mathfrak{h})$, proving the two inclusions. The commutativity of the diagram, i.e. the equality $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}\circ\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}=\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}$ is immediate from the definitions, and it implies at once the equality $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}(\mathtt{I}(\mathfrak{g}|\mathfrak{j}))=\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}\left(\mathtt{im}(\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}})\right)=\mathtt{im}(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}\circ\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}})=\mathtt{im}(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}})=\mathtt{I}(\mathfrak{g}|\mathfrak{h})$. To prove the last claim, note that in case $\mathtt{z}\in\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))$ there exists $\overline{\mathtt{z}}\in\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ such that $\mathtt{z}=\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\overline{\mathtt{z}})$, hence $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}(\mathtt{z})=\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}(\overline{\mathtt{z}})=\mathfrak{h}^{\perp}$, i.e. $\mathtt{z}\in\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$, and this shows that $\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))<\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$; as the inclusion $\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))<\mathtt{im}(\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}})=\mathtt{I}(\mathfrak{g}|\mathfrak{j})$ is immediate, we conclude that $\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))\subseteq\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$. On the other hand, $\mathtt{z}\in\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$ if there exists $\overline{\mathtt{z}}\in\mathtt{Z}(\mathfrak{g})$ such that $\mathtt{z}=\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\overline{\mathtt{z}})$ belongs to $\mathtt{Z}(\mathfrak{j}|\mathfrak{h})=\mathtt{ker}(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}})$, and this happens precisely when $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}}(\overline{\mathtt{z}})=\left(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}\circ\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}\right)(\overline{\mathtt{z}})=\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}(\mathtt{z})=\mathfrak{h}^{\perp}$, i.e. $\overline{\mathtt{z}}\in\mathtt{ker}(\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{g}})$, which implies $\mathtt{z}\in\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))$, proving finally that $\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))\supseteq\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$. □

Before the next result, let's recall the isomorphism theorems of group theory [10]: for any two subgroups A and B of an abelian group C, one has

$$A/A\cap B \cong AB/B \tag{5.2}$$

Moreover, in case $A<B$ one has

$$(C/A)/(B/A)\cong C/B \tag{5.3}$$

Lemma 2. *For* $\mathfrak{h}\subseteq\mathfrak{j}\subseteq\mathfrak{g}$ *one has isomorphisms*

$$\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) \cong \mathtt{Z}(\mathfrak{g}|\mathfrak{h})/\mathtt{Z}(\mathfrak{g}|\mathfrak{j}) \tag{5.4}$$

$$\mathtt{I}(\mathfrak{g}|\mathfrak{h}) \cong \mathtt{I}(\mathfrak{g}|\mathfrak{j})\mathtt{Z}(\mathfrak{j}|\mathfrak{h})/\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) \tag{5.5}$$

and

$$\mathtt{I}(\mathfrak{j}|\mathfrak{h})/\mathtt{I}(\mathfrak{g}|\mathfrak{h}) \cong \mathtt{Z}(\mathfrak{j})/\mathtt{I}(\mathfrak{g}|\mathfrak{j})\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) \tag{5.6}$$

Proof. Consider the restriction of $\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}$ to $\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$. By Lemma 1, its image equals $\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$, while its kernel is $\mathtt{Z}(\mathfrak{g}|\mathfrak{j})$, and Eq.(5.4) follows from the homomorphism theorem. Next, denote by ϕ the restriction of $\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}$ to $\mathtt{I}(\mathfrak{g}|\mathfrak{j})$: since, once again by Lemma 1, one has $\mathtt{im}(\phi)=\mathfrak{d}^{\mathfrak{h}}_{\mathfrak{j}}(\mathtt{I}(\mathfrak{g}|\mathfrak{j}))=\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ and $\mathtt{ker}(\phi)=\mathfrak{d}^{\mathfrak{j}}_{\mathfrak{g}}(\mathtt{Z}(\mathfrak{g}|\mathfrak{h}))=\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$, hence $\mathtt{I}(\mathfrak{g}|\mathfrak{h})\cong\mathtt{I}(\mathfrak{g}|\mathfrak{j})/\mathtt{I}(\mathfrak{g}|\mathfrak{j})\cap\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$, and Eq.(5.5) follows from this by substituting $A=\mathtt{I}(\mathfrak{g}|\mathfrak{j})$ and $B=\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$ into Eq.(5.2). Finally, Eq.(5.6) follows from Eq.(5.5) by substituting $A=\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$, $B=\mathtt{I}(\mathfrak{g}|\mathfrak{j})\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$ and $C=\mathtt{Z}(\mathfrak{j})$ into Eq.(5.3)

$$\mathtt{I}(\mathfrak{j}|\mathfrak{h})/\mathtt{I}(\mathfrak{g}|\mathfrak{h})\cong(\mathtt{Z}(\mathfrak{j})/\mathtt{Z}(\mathfrak{j}|\mathfrak{h}))/(\mathtt{I}(\mathfrak{g}|\mathfrak{j})\mathtt{Z}(\mathfrak{j}|\mathfrak{h})/\mathtt{Z}(\mathfrak{j}|\mathfrak{h}))\cong\mathtt{Z}(\mathfrak{j})/\mathtt{I}(\mathfrak{g}|\mathfrak{j})\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$$

and taking into account that $\mathtt{I}(\mathfrak{j}|\mathfrak{h})\cong\mathtt{Z}(\mathfrak{j})/\mathtt{Z}(\mathfrak{j}|\mathfrak{h})$ by Eq.(5.1). □

Lemma 3. $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\cap\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)=\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\vee\mathfrak{h}_2)$ *and* $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)<\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\cap\mathfrak{h}_2)$ *in case* $\mathfrak{h}_1,\mathfrak{h}_2\subseteq\mathfrak{g}$, *and in particular* $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})=\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ *and* $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})=\mathtt{Z}(\mathfrak{g})$ *for any* $\mathfrak{h}\subseteq\mathfrak{g}$.

Proof. $\mathtt{z}\in\mathtt{Z}(\mathfrak{g})$ belongs to $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\cap\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)$ if it is contained in both ${\mathfrak{h}_1}^{\perp}$ and ${\mathfrak{h}_2}^{\perp}$, i.e. $\mathtt{z}\subseteq{\mathfrak{h}_1}^{\perp}\cap{\mathfrak{h}_2}^{\perp}=(\mathfrak{h}_1\vee\mathfrak{h}_2)^{\perp}$, proving the first claim. But according to Lemma 2 the inclusions $\mathfrak{h}_1\cap\mathfrak{h}_2\subseteq\mathfrak{h}_1,\mathfrak{h}_2\subseteq\mathfrak{g}$ imply that both $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)$ and $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)$ are subgroups of $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\cap\mathfrak{h}_2)$, hence $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)<\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\cap\mathfrak{h}_2)$. The final claim follows from the above by substituting $\mathfrak{h}_1=\mathfrak{h}$ and $\mathfrak{h}_2=\boldsymbol{\partial}\mathfrak{g}$, and taking into account $\mathtt{Z}(\mathfrak{g}|\boldsymbol{\partial}\mathfrak{g})=\mathtt{Z}(\mathfrak{g})$. □

6. The Galois correspondence

Let's introduce the notation $\mathfrak{h}\propto\mathfrak{g}$ to indicate that $\mathfrak{h}\in\mathscr{L}$ is a central quotient of $\mathfrak{g}\in\mathscr{L}$ (equivalently, that $\mathfrak{g}$ is a central extension of $\mathfrak{h}$); in other words, $\mathfrak{h}\propto\mathfrak{g}$ means $\boldsymbol{\partial}\mathfrak{g}\subseteq\mathfrak{h}\subseteq\mathfrak{g}$ or, what is the same, $\mathfrak{h}\subseteq\mathfrak{g}\subseteq\mathfrak{h}^{\mathsf{T}}$ (c.f. Section 3). Note that this is not a transitive relation, i.e. $\mathfrak{h}\propto\mathfrak{j}$ and $\mathfrak{j}\propto\mathfrak{g}$ does not necessarily imply $\mathfrak{h}\propto\mathfrak{g}$. On the other hand, because every central quotient is a central extension of the maximal central quotient, $\mathfrak{h}\propto\mathfrak{g}$ implies $\boldsymbol{\partial}\mathfrak{g}\propto\mathfrak{h}$, and this gives in turn $\boldsymbol{\partial}\mathfrak{h}\propto\boldsymbol{\partial}\mathfrak{g}$.

Theorem 2. $\cup\mathtt{Z}(\mathfrak{g}|\mathfrak{h})=\mathfrak{h}^{\perp}$ *and* $\cup\mathtt{I}(\mathfrak{g}|\mathfrak{h})=(\boldsymbol{\partial}\mathfrak{g})^{\perp}$ *in case* $\mathfrak{h}\propto\mathfrak{g}$, *hence* $\mathfrak{h}=\mathfrak{g}/\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ *and* $\mathfrak{h}/\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\boldsymbol{\partial}\mathfrak{g}$; *in particular,*

$$\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g}) \tag{6.1}$$

Proof. If $\mathfrak{h}$ is a central quotient of $\mathfrak{g}$, then $\mathfrak{h}=\mathfrak{g}/G$ for some $G<\mathtt{Z}(\mathfrak{g})$ with $\cup G=\mathfrak{h}^{\perp}$, c.f. Section 3, hence all $\mathfrak{g}$-classes contained in $\mathfrak{h}^{\perp}$ are central and $\cup\mathtt{Z}(\mathfrak{g}|\mathfrak{h})=\mathfrak{h}^{\perp}$, consequently $G=\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$. But $\mathfrak{h}\propto\mathfrak{g}$ implies $\boldsymbol{\partial}\mathfrak{g}\propto\mathfrak{h}$, hence $\boldsymbol{\partial}\mathfrak{g}=\mathfrak{h}/H$ for some subgroup $H<\mathtt{Z}(\mathfrak{h})$, and because $\cup H=(\boldsymbol{\partial}\mathfrak{g})^{\perp}$, every $\mathfrak{h}$-class contained in $(\boldsymbol{\partial}\mathfrak{g})^{\perp}$ is central; since $(\boldsymbol{\partial}\mathfrak{g})^{\perp}$ is a union of $\mathfrak{h}$-classes, we get that $\cup\mathtt{I}(\mathfrak{g}|\mathfrak{h})=(\boldsymbol{\partial}\mathfrak{g})^{\perp}=\cup\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g})$, proving that indeed $\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g})$. □

To better grasp the meaning of Theorem 2, let's rewrite it in a (perhaps) more suggestive form. To this end, consider a subgroup $H < \mathtt{Z}(\mathfrak{g})$, and let $\mathfrak{h} = \mathfrak{g}/H$, so that $\mathfrak{h} \propto \mathfrak{g}$. Since $\mathfrak{h} \propto \mathfrak{g}$ implies $\boldsymbol{\partial}\mathfrak{h} \propto \boldsymbol{\partial}\mathfrak{g}$, there exists a subgroup $\boldsymbol{\partial} H < \mathtt{Z}(\boldsymbol{\partial}\mathfrak{g})$ such that $\boldsymbol{\partial}\mathfrak{h} = \boldsymbol{\partial}\mathfrak{g}/\boldsymbol{\partial} H$, and by Theorem 2 one has $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}) = H$ and $\mathtt{Z}(\boldsymbol{\partial}\mathfrak{g}|\boldsymbol{\partial}\mathfrak{h}) = \boldsymbol{\partial} H$. It follows from Eq.(5.1) that there is a short exact sequence

$$1 \longrightarrow \mathtt{Z}(\mathfrak{g}|\mathfrak{h}) \longrightarrow \mathtt{Z}(\mathfrak{g}) \longrightarrow \mathtt{I}(\mathfrak{g}|\mathfrak{h}) \longrightarrow 1 \tag{6.2}$$

As $\mathfrak{h} \propto \mathfrak{g}$ implies $\boldsymbol{\partial}\mathfrak{g} \propto \mathfrak{h}$, we can substitute $\boldsymbol{\partial}\mathfrak{g}$ for $\mathfrak{h}$, and simultaneously $\mathfrak{h}$ for $\mathfrak{g}$ in Eq.(6.2) to yield the exact sequence

$$1 \longrightarrow \mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g}) \longrightarrow \mathtt{Z}(\mathfrak{h}) \longrightarrow \mathtt{I}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g}) \longrightarrow 1 \tag{6.3}$$

But $\mathtt{I}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g}) = \mathtt{Z}(\boldsymbol{\partial}\mathfrak{g}|\boldsymbol{\partial}\mathfrak{h}) = \boldsymbol{\partial} H$ and $\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g}) = \mathtt{I}(\mathfrak{g}|\mathfrak{h}) \cong \mathtt{Z}(\mathfrak{g})/\mathtt{Z}(\mathfrak{g}|\mathfrak{h}) = \mathtt{Z}(\mathfrak{g})/H$ by Theorem 2 and Eq.(5.1), hence we arrive finally at the exact sequence

$$1 \longrightarrow \mathtt{Z}(\mathfrak{g})/H \longrightarrow \mathtt{Z}(\mathfrak{g}/H) \longrightarrow \boldsymbol{\partial} H \longrightarrow 1 \tag{6.4}$$

which shows clearly that $\boldsymbol{\partial} H$ measures the extent by which $\mathtt{Z}(\mathfrak{g}/H)$ differs from the naive expectation $\mathtt{Z}(\mathfrak{g})/H$. In particular, one has $\mathtt{Z}(\mathfrak{g}/H) \cong \mathtt{Z}(\mathfrak{g})/H$ only in case $\boldsymbol{\partial} H = 1$, i.e. when $\boldsymbol{\partial}\mathfrak{g} = \boldsymbol{\partial}\mathfrak{h}$.

Lemma 4. *If $\mathfrak{h} \propto \mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{j} \subseteq \mathfrak{g}$, then $\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) < \mathtt{I}(\mathfrak{g}|\mathfrak{j})$, consequently*

$$\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) \cong \mathtt{Z}(\mathfrak{g}|\mathfrak{h})/\mathtt{Z}(\mathfrak{g}|\mathfrak{j}) \tag{6.5}$$

and

$$\mathtt{Z}(\boldsymbol{\partial}\mathfrak{g}|\boldsymbol{\partial}\mathfrak{j}) \cong \mathtt{I}(\mathfrak{j}|\mathfrak{h})/\mathtt{I}(\mathfrak{g}|\mathfrak{h}) \tag{6.6}$$

Proof. $\mathfrak{h} \propto \mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{j} \subseteq \mathfrak{g}$ imply $\mathfrak{j} \propto \mathfrak{g}$, hence $\boldsymbol{\partial}\mathfrak{g} \propto \mathfrak{j}$, consequently $\mathtt{I}(\mathfrak{g}|\mathfrak{j}) = \mathtt{Z}(\mathfrak{j}|\boldsymbol{\partial}\mathfrak{g})$ and $\mathtt{I}(\mathfrak{j}|\boldsymbol{\partial}\mathfrak{g}) = \mathtt{Z}(\boldsymbol{\partial}\mathfrak{g}|\boldsymbol{\partial}\mathfrak{j})$ according to Theorem 2. On the other hand, $\boldsymbol{\partial}\mathfrak{g} \subseteq \mathfrak{h} \subseteq \mathfrak{j}$ gives $\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) < \mathtt{Z}(\mathfrak{j}|\boldsymbol{\partial}\mathfrak{g})$ according to Lemma 1, hence $\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) < \mathtt{Z}(\mathfrak{j}|\boldsymbol{\partial}\mathfrak{g}) = \mathtt{I}(\mathfrak{g}|\mathfrak{j})$ by the above. But $\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) < \mathtt{I}(\mathfrak{g}|\mathfrak{j})$ implies $\mathtt{I}(\mathfrak{g}|\mathfrak{j}) \cap \mathtt{Z}(\mathfrak{j}|\mathfrak{h}) = \mathtt{Z}(\mathfrak{j}|\mathfrak{h})$ and $\mathtt{I}(\mathfrak{g}|\mathfrak{j})\mathtt{Z}(\mathfrak{j}|\mathfrak{h}) = \mathtt{I}(\mathfrak{g}|\mathfrak{j})$, and taking this into account, Eq.(5.4) reduces to Eq.(6.5), while Eq.(5.6) leads to

$$\mathtt{I}(\mathfrak{j}|\mathfrak{h})/\mathtt{I}(\mathfrak{g}|\mathfrak{h}) \cong \mathtt{Z}(\mathfrak{j})/\mathtt{I}(\mathfrak{g}|\mathfrak{j}) = \mathtt{Z}(\mathfrak{j})/\mathtt{Z}(\mathfrak{j}|\boldsymbol{\partial}\mathfrak{g}) \cong \mathtt{I}(\mathfrak{j}|\boldsymbol{\partial}\mathfrak{g}) = \mathtt{Z}(\boldsymbol{\partial}\mathfrak{g}|\boldsymbol{\partial}\mathfrak{j})$$

in view of Theorem 2 and Eq.(5.1), proving Eq.(6.6). ☐

For subgroups $H_1 < H_2 < \mathtt{Z}(\mathfrak{g})$, Eq.(6.5) with the choice $\mathfrak{j} = \mathfrak{g}/H_1$ and $\mathfrak{h} = \mathfrak{g}/H_2$ gives us an isomorphism

$$\mathtt{Z}(\mathfrak{g}/H_1|\mathfrak{g}/H_2) \cong H_2/H_1 \tag{6.7}$$

while combining Eq.(6.6) with Eq.(6.2) leads to the exact sequence

$$1 \longrightarrow \mathtt{Z}(\mathfrak{g})/H_2 \longrightarrow \mathtt{I}(\mathfrak{g}/H_1|\mathfrak{g}/H_2) \longrightarrow \boldsymbol{\partial} H_1 \longrightarrow 1 \tag{6.8}$$

To formulate our next result, let's introduce the notation $\mathscr{L}(\mathfrak{h},\mathfrak{g}) = \{\mathfrak{j} \in \mathscr{L} \mid \mathfrak{h} \subseteq \mathfrak{j} \subseteq \mathfrak{g}\}$ for $\mathfrak{g}, \mathfrak{h} \in \mathscr{L}$; notice that $\mathscr{L}(\mathfrak{h},\mathfrak{g})$ is empty unless $\mathfrak{h} \subseteq \mathfrak{g}$. In particular, $\mathscr{L}(\boldsymbol{\partial}\mathfrak{g},\mathfrak{g})$ equals the collection of central quotients, and $\mathscr{L}(\mathfrak{g},\mathfrak{g}^{\mathsf{T}})$ that of central extensions of $\mathfrak{g}$ according to the comments following Eq.(3.3). As a byproduct of Theorem 2, one gets an interesting extension of the Galois correspondence that has been described in Section 3, according to which there is, in case $\mathfrak{h}$ is a central quotient of $\mathfrak{g}$, a one-to-one correspondence between the subgroups of $\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ and those central quotients of $\mathfrak{g}$ that contain $\mathfrak{h}$. But $\mathfrak{h} \propto \mathfrak{g}$ implies $\boldsymbol{\partial}\mathfrak{g} \propto \mathfrak{h}$, i.e. $\boldsymbol{\partial}\mathfrak{g}$ is a central quotient of $\mathfrak{h}$, hence there is a one-to-one correspondence between the subgroups of $\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g}) = \mathtt{I}(\mathfrak{g}|\mathfrak{h})$ and those central quotients of $\mathfrak{h}$ that contain $\boldsymbol{\partial}\mathfrak{g}$. Since these are nothing but the central quotients of $\mathfrak{g}$ contained in $\mathfrak{h}$, in the end we get a one-to-one correspondence between the latter and the subgroups of $\mathtt{I}(\mathfrak{g}|\mathfrak{h})$. For later reference, let's formulate this result as a Lemma.

Lemma 5. *In case* $\mathfrak{h}\propto\mathfrak{g}$*, there are one-to-one correspondences between* $\mathscr{L}(\mathfrak{h},\mathfrak{g})$ *and subgroups of* $\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ *on one hand, and between* $\mathscr{L}(\boldsymbol{\partial}\mathfrak{g},\mathfrak{h})$ *and subgroups of* $\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ *on the other.*

Combined with the modularity of the lattice $\mathscr{L}$, Lemma 5 leads to important results, notably the following strengthening of Lemma 3.

Lemma 6. $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)=\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\cap\mathfrak{h}_2)$ *if both* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *are central quotients of* $\mathfrak{g}$*.*

Proof. Indeed, the inclusions $\boldsymbol{\partial}\mathfrak{g}\subseteq\mathfrak{h}_1,\mathfrak{h}_2\subseteq\mathfrak{g}$ imply $\boldsymbol{\partial}\mathfrak{h}_1\subseteq\boldsymbol{\partial}\mathfrak{g}\subseteq\mathfrak{h}_1\cap\mathfrak{h}_2\subseteq\mathfrak{h}_1$ and $\boldsymbol{\partial}(\mathfrak{h}_1\vee\mathfrak{h}_2)\subseteq\boldsymbol{\partial}\mathfrak{g}\subseteq\mathfrak{h}_2\subseteq\mathfrak{h}_1\vee\mathfrak{h}_2$, hence $\mathfrak{h}_1\cap\mathfrak{h}_2\propto\mathfrak{h}_1$ and $\mathfrak{h}_2\propto\mathfrak{h}_1\vee\mathfrak{h}_2$, so there is, by Lemma 5, a one-to-one correspondence between $\mathscr{L}(\mathfrak{h}_2,\mathfrak{h}_1\vee\mathfrak{h}_2)$ and subgroups of $\mathtt{Z}(\mathfrak{h}_1\vee\mathfrak{h}_2|\mathfrak{h}_2)$ on one hand, and between $\mathscr{L}(\mathfrak{h}_1\cap\mathfrak{h}_2,\mathfrak{h}_1)$ and subgroups of $\mathtt{Z}(\mathfrak{h}_1|\mathfrak{h}_1\cap\mathfrak{h}_2)$ on the other. $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)$ is a subgroup of $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\cap\mathfrak{h}_2)$ by Lemma 3: should it be a proper subgroup, the factor group $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)/\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)$, which is isomorphic to $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)/\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\vee\mathfrak{h}_2)\cong\mathtt{Z}(\mathfrak{h}_1\vee\mathfrak{h}_2|\mathfrak{h}_2)$ according to Eq.(5.2) and Lemma 4, would have less subgroups than the factor group $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\cap\mathfrak{h}_2)/\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)\cong\mathtt{Z}(\mathfrak{h}_1|\mathfrak{h}_1\cap\mathfrak{h}_2)$, contradicting the one-to-one correspondence between $\mathscr{L}(\mathfrak{h}_2,\mathfrak{h}_1\vee\mathfrak{h}_2)$ and $\mathscr{L}(\mathfrak{h}_1\cap\mathfrak{h}_2,\mathfrak{h}_1)$ that follows from the modularity of $\mathscr{L}$. □

The above result has the following important consequence: consider subgroups $H_1,H_2<\mathtt{Z}(\mathfrak{g})$, and let $\mathfrak{h}_1=\mathfrak{g}/H_1$ and $\mathfrak{h}_2=\mathfrak{g}/H_2$, so that $H_1=\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1)$ and $H_2=\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_2)$. Since $H_1\cap H_2=\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\vee\mathfrak{h}_2)$ by Lemma 3, while $H_1H_2=\mathtt{Z}(\mathfrak{g}|\mathfrak{h}_1\cap\mathfrak{h}_2)$ according to Lemma 6, one has by Theorem 2

$$(\mathfrak{g}/H_1)\vee(\mathfrak{g}/H_2)=\mathfrak{g}/(H_1\cap H_2) \tag{6.9}$$

and

$$(\mathfrak{g}/H_1)\cap(\mathfrak{g}/H_2)=\mathfrak{g}/(H_1H_2) \tag{6.10}$$

which means that the correspondences of Lemma 5 are actually lattice isomorphisms.

In the same vein, one gets the following far-reaching generalization of Theorem 2.

Theorem 3. $\mathfrak{g}/\mathtt{Z}(\mathfrak{g}|\mathfrak{h})=\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g}$ *and* $\mathfrak{h}/\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g}$ *for* $\mathfrak{h}\subseteq\mathfrak{g}$*, and in particular*

$$\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g}) \tag{6.11}$$

Proof. As $\mathfrak{h}\subseteq\mathfrak{g}$ implies both $\boldsymbol{\partial}\mathfrak{g}\subseteq\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g}\subseteq\mathfrak{g}$ and $\boldsymbol{\partial}\mathfrak{h}\subseteq\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g}\subseteq\mathfrak{h}$, we have $\mathfrak{g}/\mathtt{Z}(\mathfrak{g}|\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})=\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g}$ and $\mathfrak{h}/\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})=\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g}$ according to Theorem 2. But $\mathtt{Z}(\mathfrak{g}|\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})=\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ by Lemma 3, proving the first assertion, while the second one would follow from the equality $\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})$. To prove the latter, consider $\mathfrak{j}=\mathfrak{h}/\mathtt{I}(\mathfrak{g}|\mathfrak{h})$; we claim that $\mathfrak{j}\subseteq\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g}$. The inclusion $\mathfrak{j}\subseteq\mathfrak{h}$ is obvious, while $\mathfrak{j}\subseteq\boldsymbol{\partial}\mathfrak{g}$ follows from the observation that, because every central $\mathfrak{g}$-class is contained in a central $\mathfrak{j}$-class, the union $\cup\mathtt{Z}(\mathfrak{g})=(\boldsymbol{\partial}\mathfrak{g})^{\perp}$ of all central $\mathfrak{g}$-classes should be contained in the union $\cup\mathtt{I}(\mathfrak{g}|\mathfrak{j})=\mathfrak{j}^{\perp}$ of all those central $\mathfrak{j}$-classes that contain at least one central $\mathfrak{g}$-class, hence $(\boldsymbol{\partial}\mathfrak{g})^{\perp}\subseteq\mathfrak{j}^{\perp}$ or, what is the same, $\mathfrak{j}\subseteq\boldsymbol{\partial}\mathfrak{g}$. But $\mathfrak{j}\subseteq\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g}$ means that $\mathfrak{h}/\mathtt{I}(\mathfrak{g}|\mathfrak{h})\subseteq\mathfrak{h}/\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})$, hence $\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})$ is a subgroup of $\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ by Lemma 5. We claim that they are actually equal.

According to Lemma 5, there is a one-to-one correspondence between subgroups of $\mathtt{I}(\mathfrak{g}|\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})$ and $\mathscr{L}(\boldsymbol{\partial}\mathfrak{g},\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})$ on one hand, and between subgroups of $\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})$ and $\mathscr{L}(\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g},\mathfrak{h})$ on the other. But $\mathtt{I}(\mathfrak{g}|\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})\cong\mathtt{Z}(\mathfrak{g})/\mathtt{Z}(\mathfrak{g}|\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})=\mathtt{Z}(\mathfrak{g})/\mathtt{Z}(\mathfrak{g}|\mathfrak{h})\cong\mathtt{I}(\mathfrak{g}|\mathfrak{h})$ by Eq.(5.1): should $\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})$ be a proper subgroup of $\mathtt{I}(\mathfrak{g}|\mathfrak{h})$, it would have less subgroups, contradicting the one-to-one correspondence between $\mathscr{L}(\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g},\mathfrak{h})$ and $\mathscr{L}(\boldsymbol{\partial}\mathfrak{g},\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})$ that follows from the modularity of the deconstruction lattice. □

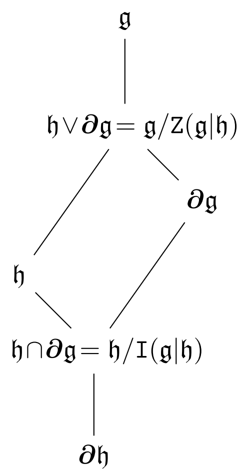

Fig.3: Inclusion relations relevant to Theorem 3.

7. A long exact sequence

An interesting addendum to the previous results follows from the observation that, thanks to Eq.(6.1), the short exact sequences Eq.(6.2) and Eq.(6.3) can be combined for $\mathfrak{h}\propto\mathfrak{g}$ into a four-term exact sequence, and the latter leads, by means of a recursive process, to a long exact sequence connecting the centers of higher central quotients.

Too see how this come about, remember that $\mathtt{I}(\mathfrak{g}|\mathfrak{h})=\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g})$ for $\mathfrak{h}\propto\mathfrak{g}$ by Theorem 2, hence the short exact sequence of Eq.(6.2) reads in this case

$$1\longrightarrow\mathtt{Z}(\mathfrak{g}|\mathfrak{h})\longrightarrow\mathtt{Z}(\mathfrak{g})\longrightarrow\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g})\longrightarrow 1 \tag{7.1}$$

Next, since $\mathfrak{h}\propto\mathfrak{g}$ implies $\boldsymbol{\partial}\mathfrak{g}\propto\mathfrak{h}$, we can substitute $\boldsymbol{\partial}\mathfrak{g}$ for $\mathfrak{h}$, and simultaneously $\mathfrak{h}$ for $\mathfrak{g}$ in Eq.(7.1) to yield

$$1\longrightarrow\mathtt{Z}(\mathfrak{h}|\boldsymbol{\partial}\mathfrak{g})\longrightarrow\mathtt{Z}(\mathfrak{h})\longrightarrow\mathtt{Z}(\boldsymbol{\partial}\mathfrak{g}|\boldsymbol{\partial}\mathfrak{h})\longrightarrow 1 \tag{7.2}$$

and combining this last sequence with Eq.(7.1) gives the four-term exact sequence

$$1\longrightarrow\mathtt{Z}(\mathfrak{g}|\mathfrak{h})\longrightarrow\mathtt{Z}(\mathfrak{g})\longrightarrow\mathtt{Z}(\mathfrak{h})\longrightarrow\mathtt{Z}(\boldsymbol{\partial}\mathfrak{g}|\boldsymbol{\partial}\mathfrak{h})\longrightarrow 1 \tag{7.3}$$

But there is no reason to stop here, as $\mathfrak{h}\propto\mathfrak{g}$ implies $\boldsymbol{\partial}^k\mathfrak{h}\propto\boldsymbol{\partial}^k\mathfrak{g}$ for any $k\geq 1$, so we can substitute $\boldsymbol{\partial}^k\mathfrak{g}$ for $\mathfrak{g}$ and $\boldsymbol{\partial}^k\mathfrak{h}$ for $\mathfrak{h}$ in Eq.(7.3) to obtain an exact sequence

$$1\longrightarrow\mathtt{Z}(\boldsymbol{\partial}^k\mathfrak{g}|\boldsymbol{\partial}^k\mathfrak{h})\longrightarrow\mathtt{Z}(\boldsymbol{\partial}^k\mathfrak{g})\longrightarrow\mathtt{Z}(\boldsymbol{\partial}^k\mathfrak{h})\longrightarrow\mathtt{Z}(\boldsymbol{\partial}^{k+1}\mathfrak{g}|\boldsymbol{\partial}^{k+1}\mathfrak{h})\longrightarrow 1$$

Combining Eq.(7.3) with the above sequences for $k=1,\ldots,n$ leads finally to the long exact sequence

$$1\rightarrow\mathtt{Z}(\mathfrak{g}|\mathfrak{h})\longrightarrow\mathtt{Z}(\mathfrak{g})\longrightarrow\mathtt{Z}(\mathfrak{h})\longrightarrow\mathtt{Z}(\boldsymbol{\partial}\mathfrak{g})\longrightarrow\mathtt{Z}(\boldsymbol{\partial}\mathfrak{h})\longrightarrow\cdots\longrightarrow\mathtt{Z}(\boldsymbol{\partial}^n\mathfrak{g})\rightarrow\mathtt{Z}(\boldsymbol{\partial}^n\mathfrak{h})\rightarrow\mathtt{Z}(\boldsymbol{\partial}^{n+1}\mathfrak{g}|\boldsymbol{\partial}^{n+1}\mathfrak{h})\longrightarrow 1$$

Since $\boldsymbol{\partial}^{n+1}\mathfrak{g}=\boldsymbol{\partial}^n\mathfrak{g}$ for large enough n (c.f. Section 3) and $\boldsymbol{\partial}^{k+1}\mathfrak{g}\subseteq\boldsymbol{\partial}^k\mathfrak{h}\subseteq\boldsymbol{\partial}^k\mathfrak{g}$ for $\mathfrak{h}\propto\mathfrak{g}$ and $k>1$, there exists a largest integer N such that $\boldsymbol{\partial}^N\mathfrak{h}\neq\boldsymbol{\partial}^N\mathfrak{g}$ (clearly, this integer depends on both $\mathfrak{g}$ and $\mathfrak{h}$). This leads finally to the following result.

Theorem 4. *If* $\mathfrak{h}\propto\mathfrak{g}$ *and* N *is the largest integer such that* $\boldsymbol{\partial}^N\mathfrak{h}\neq\boldsymbol{\partial}^N\mathfrak{g}$*, then there is a long exact sequence*

$$1\longrightarrow \mathtt{Z}(\mathfrak{g}|\mathfrak{h})\longrightarrow \mathtt{Z}(\mathfrak{g})\longrightarrow \mathtt{Z}(\mathfrak{h})\longrightarrow \mathtt{Z}(\boldsymbol{\partial}\mathfrak{g})\longrightarrow$$
$$\longrightarrow \mathtt{Z}(\boldsymbol{\partial}\mathfrak{h})\longrightarrow\cdots\longrightarrow \mathtt{Z}(\boldsymbol{\partial}^N\mathfrak{g})\longrightarrow \mathtt{Z}(\boldsymbol{\partial}^N\mathfrak{h})\longrightarrow 1$$

Proof. This is a direct consequence of the above considerations, taking into account that $\mathtt{Z}(\boldsymbol{\partial}^{N+1}\mathfrak{g}|\boldsymbol{\partial}^{N+1}\mathfrak{h})=1$ since $\boldsymbol{\partial}^{N+1}\mathfrak{h}=\boldsymbol{\partial}^{N+1}\mathfrak{g}$. □

8. Extensions vs quotients

We have gone a long way in achieving our original goal of understanding the relations between the centers of different elements of $\mathscr{L}$, c.f. the exact sequences Eq.(6.4) and Eq.(6.8), or the isomorphism Eq.(6.7). But this is by no means the end of the story, as practical considerations suggest that one should also consider the case of central extensions besides that of quotients. Of course, since $\mathfrak{h}\propto\mathfrak{g}$ not only means that $\mathfrak{h}$ is a central quotient of $\mathfrak{g}$, but also that $\mathfrak{g}$ is a central extension of $\mathfrak{h}$, all the results of Section 6 apply, but they have the drawback of referring to central quotients of $\mathfrak{g}$ and $\mathfrak{h}$ instead of their central extensions. Fortunately, this can be remedied by using the following result.

Lemma 7. $\mathtt{Z}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)\cong\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ *and* $\mathtt{I}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)\cong\mathtt{Z}(\mathfrak{h}^\intercal|\mathfrak{g})$ *whenever* $\mathfrak{h}\propto\mathfrak{g}$.

Proof. First, note that $\mathfrak{h}\propto\mathfrak{g}$ iff $\mathfrak{g}^\perp\propto\mathfrak{h}^\perp$. Since by assumption, $\mathfrak{h}=\mathfrak{g}/H$ for some subgroup $H<\mathtt{Z}(\mathfrak{g})$, hence [6] there is a subgroup $H^\perp$ of $\mathtt{Z}(\mathfrak{h}^\perp)$ isomorphic with H such that $\mathfrak{g}^\perp=\mathfrak{h}^\perp/H^\perp$, and this implies the isomorphism $\mathtt{Z}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)=H^\perp\cong H=\mathtt{Z}(\mathfrak{g}|\mathfrak{h})$ by Theorem 2. As to the second assertion, it followsEq.(6.1) and the above since $\mathtt{I}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)=\mathtt{Z}(\mathfrak{g}^\perp|\boldsymbol{\partial}(\mathfrak{h}^\perp))=\mathtt{Z}(\mathfrak{g}^\perp|(\mathfrak{h}^\intercal)^\perp)\cong\mathtt{Z}(\mathfrak{h}^\intercal|\mathfrak{g})$, proving the claim. □

In particular, substituting $\mathfrak{h}^\perp$ for $\mathfrak{g}$ and $\mathfrak{g}^\perp$ for $\mathfrak{h}$ in Eq.(6.2), and making use of Lemma 7 leads to the short exact sequence

$$1\longrightarrow\mathtt{Z}(\mathfrak{g}|\mathfrak{h})\longrightarrow\mathtt{Z}(\mathfrak{h}^\perp)\longrightarrow\mathtt{Z}(\mathfrak{h}^\intercal|\mathfrak{g})\longrightarrow 1 \tag{8.1}$$

to be contrasted with Eq.(7.1), while a similar argument applied to Eq.(7.3) leads to the exact sequence

$$1\longrightarrow\mathtt{Z}(\mathfrak{g}|\mathfrak{h})\longrightarrow\mathtt{Z}(\mathfrak{h}^\perp)\longrightarrow\mathtt{Z}(\mathfrak{g}^\perp)\longrightarrow\mathtt{Z}(\mathfrak{g}^\intercal|\mathfrak{h}^\intercal)\longrightarrow 1 \tag{8.2}$$

Let's note that, by making use of Theorem 3, one can drop the requirement $\mathfrak{h}\propto\mathfrak{g}$, resulting in the following generalization of Lemma 7.

Lemma 8. $\mathtt{Z}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)\cong\mathtt{Z}(\mathfrak{g}\cap\mathfrak{h}^\intercal|\mathfrak{h})$ *and* $\mathtt{I}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)\cong\mathtt{Z}(\mathfrak{h}^\intercal\vee\mathfrak{g}|\mathfrak{g})$ *for* $\mathfrak{h}\subseteq\mathfrak{g}$.

Proof. First of all, let's note that $\boldsymbol{\partial}(\mathfrak{h}^\perp)\subseteq\mathfrak{g}^\perp\cap\boldsymbol{\partial}(\mathfrak{h}^\perp)\subseteq\mathfrak{g}^\perp\subseteq\mathfrak{g}^\perp\vee\boldsymbol{\partial}(\mathfrak{h}^\perp)\subseteq\mathfrak{h}^\perp$ whenever $\mathfrak{h}\subseteq\mathfrak{g}$, and this implies $\mathtt{Z}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)=\mathtt{Z}(\mathfrak{h}^\perp|\mathfrak{g}^\perp\vee\boldsymbol{\partial}(\mathfrak{h}^\perp))\cong\mathtt{Z}(\mathfrak{g}\cap\mathfrak{h}^\intercal|\mathfrak{h})$ and $\mathtt{I}(\mathfrak{h}^\perp|\mathfrak{g}^\perp)=\mathtt{Z}(\mathfrak{g}^\perp|\mathfrak{g}^\perp\cap\boldsymbol{\partial}(\mathfrak{h}^\perp))\cong\mathtt{Z}(\mathfrak{g}\vee\mathfrak{h}^\intercal|\mathfrak{g})$ according to Theorem 3 and Lemma 7. □

Substituting $\mathfrak{h}^\perp$ for $\mathfrak{g}$ and $\mathfrak{g}^\perp$ for $\mathfrak{h}$ in Eq.(6.2) gives (for $\mathfrak{h}\subseteq\mathfrak{g}$)

$$1\longrightarrow\mathtt{Z}(\mathfrak{g}\cap\mathfrak{h}^\intercal|\mathfrak{h})\longrightarrow\mathtt{Z}(\mathfrak{h}^\perp)\longrightarrow\mathtt{Z}(\mathfrak{h}^\intercal\vee\mathfrak{g}|\mathfrak{g})\longrightarrow 1 \tag{8.3}$$

when taking into account Lemma 8, to be compared to the sequence

$$1\longrightarrow\mathtt{Z}(\mathfrak{g}|\mathfrak{h}\vee\boldsymbol{\partial}\mathfrak{g})\longrightarrow\mathtt{Z}(\mathfrak{g})\longrightarrow\mathtt{Z}(\mathfrak{h}|\mathfrak{h}\cap\boldsymbol{\partial}\mathfrak{g})\longrightarrow 1 \tag{8.4}$$

that follows from Eq.(6.11).

9. Summary and outlook

The main theme of our investigations was to characterize the mutual relations between the centers of different elements of the deconstruction lattice $\mathscr{L}$, with particular emphasis on the central quotients and extensions of a given $\mathfrak{g}\in\mathscr{L}$. We have seen that one can go a long way in this direction using exact sequences, as exhibited by results like Eq.(6.4), Theorem 4 and Eq.(8.2). While not providing a direct and constructive description of the relevant groups, this information is usually enough to pin down (more or less uniquely) their structure, and this is what matters for most applications. Once the relevant centers are known, one can use this knowledge to simplify greatly otherwise cumbersome computations that could prove difficult in case of large examples. In particular, the explicit Galois correspondence described by Lemma 5 is a very useful tool in actual computations, while Lemma 8 settles the dual relation of central quotients and extensions.

Apart from the practical considerations set forth above, there are more conceptual issues underlying the interest in the previous results. One such is the search for an analogue of the famous lemma of Grün [10], according to which the upper central series of a perfect group has length at most two. In our context this would mean that, provided $\mathfrak{g}\in\mathscr{L}$ does not contain any simple current, the maximal central quotient $\boldsymbol{\partial}\mathfrak{g}$ should be centerless, i.e. $\mathtt{Z}(\boldsymbol{\partial}\mathfrak{g})=1$. While this holds automatically for local $\mathfrak{g}\in\mathscr{L}$ corresponding to some finite group, the generic case seems much more difficult to prove, as there is no clear adaptation of the group theoretic techniques used in the proof of Grün's lemma.

Another interesting question concerns the analogue of Ito's theorem [22, 23], which in our case is tantamount to the claim that the ratio

$$\frac{1}{\mathtt{d}_\alpha}\sum_{p\in\boldsymbol{\partial}\mathfrak{g}}\mathtt{d}_p^2$$

is an algebraic integer for every $\alpha\in\mathfrak{g}$. Should this claim hold, it would restrict severely the arithmetic properties of the quantum dimensions and, more generally, the Galois action on the primaries [6]. As before, the claim follows from known group theoretic results for local $\mathfrak{g}\in\mathscr{L}$ corresponding to some finite group, but its generalization to arbitrary $\mathfrak{g}\in\mathscr{L}$ is far from being obvious.

Finally, we should note that in the preceding discussions we have neglected one important aspect, namely that the center $\mathtt{Z}(\mathfrak{g})$ of any given $\mathfrak{g}\in\mathscr{L}$ is not simply an abelian group, but has a natural permutation action on the set of $\mathfrak{g}$-classes [6], which is compatible with the restriction map. This permutation action is far from being arbitrary, as can be seen most directly on the example of abelian $\mathfrak{g}$, i.e. when all primaries in $\mathfrak{g}$ are simple currents, since in this case the $\mathfrak{g}$-classes are in one-to-one correspondence with the characters of $\mathtt{Z}(\mathfrak{g})$, and the permutation action is regular, i.e. the action of $\mathtt{Z}(\mathfrak{g})$ on itself by translations. On the other extreme, if $\mathfrak{g}$ equals the maximal element of the deconstruction lattice $\mathscr{L}$, then each $\mathfrak{g}$-class contains precisely one primary, with central classes containing the simple currents, and the corresponding permutation action is nothing but the action of the group of simple currents on the set of primaries that is induced by the fusion product. This latter action is known to have non-trivial properties [24]. It is natural to expect that for $\mathfrak{g}\in\mathscr{L}$ intermediate between these two extremes, the permutation action of $\mathtt{Z}(\mathfrak{g})$ still enjoys some interesting properties, but this circle of questions lies outside the scope of the present note.

Appendix A. Proof of modularity

Lemma 9. *For $\mathfrak{g},\mathfrak{h}\in\mathscr{L}$, a primary p belongs to the join $\mathfrak{g}\vee\mathfrak{h}$ iff there exists $\alpha\in\mathfrak{g}$ and $\beta\in\mathfrak{h}$ such that $N_{\alpha\beta}^{p}>0$.*

Proof. Let $\mathfrak{g}\sqcup\mathfrak{h}=\left\{p\,|\,N_{\alpha\beta}^{p}>0 \text{ for some } \alpha\in\mathfrak{g} \text{ and } \beta\in\mathfrak{h}\right\}$. Clearly, both $\mathfrak{g}$ and $\mathfrak{h}$ are contained in $\mathfrak{g}\sqcup\mathfrak{h}$. On the other hand, $p\in\mathfrak{g}\sqcup\mathfrak{h}$ means that there exists $\alpha\in\mathfrak{g}$ and $\beta\in\mathfrak{h}$ such that $N_{\alpha\beta}^{p}>0$: since $\mathfrak{g}\vee\mathfrak{h}$ is the least element of $\mathscr{L}$ containing both $\mathfrak{g}$ and $\mathfrak{h}$, we have necessarily $p\in\mathfrak{g}\vee\mathfrak{h}$, hence $\mathfrak{g}\sqcup\mathfrak{h}\subseteq\mathfrak{g}\vee\mathfrak{h}$. To prove that they are actually equal, it suffices to show that $\mathfrak{g}\sqcup\mathfrak{h}\in\mathscr{L}$, i.e. that $p,q\in\mathfrak{g}\sqcup\mathfrak{h}$ and $N_{pq}^{r}>0$ implies $r\in\mathfrak{g}\sqcup\mathfrak{h}$. But this is tantamount to showing that, if there exists $\alpha_1,\alpha_2\in\mathfrak{g}$ and $\beta_1,\beta_2\in\mathfrak{h}$ such that $N_{\alpha_1\beta_1}^{p}>0$ and $N_{\alpha_2\beta_2}^{q}>0$, then for all r such that $N_{pq}^{r}>0$ there exists $\alpha_3\in\mathfrak{g}$ and $\beta_3\in\mathfrak{h}$ satisfying $N_{\alpha_3\beta_3}^{r}>0$. By the associativity of the fusion algebra, it follows form our assumptions that

$$0<N_{\alpha_1\beta_1}^{p}N_{\alpha_2\beta_2}^{q}N_{pq}^{r}\leq\sum_{s,t}N_{\alpha_1\beta_1}^{s}N_{\alpha_2\beta_2}^{t}N_{st}^{r}=$$

$$\sum_{w,t}N_{\alpha_1 t}^{w}N_{\beta_1 w}^{r}N_{\alpha_2\beta_2}^{t}=\sum_{w,u}N_{\alpha_1\alpha_2}^{u}N_{\beta_2 u}^{w}N_{\beta_1 w}^{r}=\sum_{u,v}N_{\alpha_1\alpha_2}^{u}N_{\beta_1\beta_2}^{v}N_{uv}^{r}$$

i.e. there should exist primaries α_3 and β_3 such that $N_{\alpha_1\alpha_2}^{\alpha_3}$, $N_{\beta_1\beta_2}^{\beta_3}$ and $N_{\alpha_3\beta_3}^{r}$ are all positive. But $\mathfrak{g},\mathfrak{h}\in\mathscr{L}$ and $N_{\alpha_1\alpha_2}^{\alpha_3}N_{\beta_1\beta_2}^{\beta_3}>0$ implies that $\alpha_3\in\mathfrak{g}$ and $\beta_3\in\mathfrak{h}$, consequently $r\in\mathfrak{g}\sqcup\mathfrak{h}$. □

The previous result allows for the following streamlined proof of the modularity of $\mathscr{L}$ (recall that $N_{pq}^{\overline{r}}=N_{pr}^{\overline{q}}$, with $\overline{p}$ denoting the charge conjugate of p).

Theorem 5. *The lattice $\mathscr{L}$ is modular, that is*

$$\mathfrak{h}_2\cap(\mathfrak{h}_1\vee\mathfrak{g})\subseteq\mathfrak{h}_1\vee(\mathfrak{h}_2\cap\mathfrak{g})$$

for $\mathfrak{g},\mathfrak{h}_1,\mathfrak{h}_2\in\mathscr{L}$ such that $\mathfrak{h}_1\subseteq\mathfrak{h}_2$.

Proof. Suppose that $\alpha\in\mathfrak{h}_2\cap(\mathfrak{h}_1\vee\mathfrak{g})$. Then $\alpha\in\mathfrak{h}_2$, hence $\overline{\alpha}\in\mathfrak{h}_2$, and by Lemma 9 there exists primaries $\beta\in\mathfrak{h}_1$ and $\gamma\in\mathfrak{g}$ such that $N_{\beta\gamma}^{\alpha}>0$. But $\beta\in\mathfrak{h}_2$ since $\mathfrak{h}_1\subseteq\mathfrak{h}_2$, consequently $N_{\beta\overline{\alpha}}^{\overline{\gamma}}=N_{\beta\gamma}^{\alpha}$ implies $\overline{\gamma}\in\mathfrak{h}_2$, i.e. $\gamma\in\mathfrak{h}_2$. All in all, we get that $N_{\beta\gamma}^{\alpha}>0$ with $\beta\in\mathfrak{h}_1$ and $\gamma\in\mathfrak{g}\cap\mathfrak{h}_2$, consequently $\alpha\in\mathfrak{h}_1\vee(\mathfrak{h}_2\cap\mathfrak{g})$ by Lemma 9, proving the assertion. □

Modularity has many important consequences [9], e.g. the Kurosh-Ore theorem or having a modular rank function, but our analysis makes mainly use of the so-called diamond isomorphism theorem, according to which, for any two elements a,b of a modular lattice $(L,\wedge,\vee)$, there is an order-preserving one-to-one correspondence between the sets $\{x\in L\,|\,a\wedge b\leq x\leq a\}$ and $\{x\in L\,|\,b\leq x\leq a\vee b\}$.

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