Symmetry preserving cutoff regularization

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Abstract

A Lorentz and gauge symmetry preserving cutoff regularization method is proposed in 4 dimension. We use the conditions of gauge invariance or the freedom of shift of the loop-momentum to define how to evaluate terms carrying Lorentz indices, e.g. proportional to $k_\mu k_\nu$. All the terms are unambiguous and the finite results agree with dimensional regularization.

Introduction

- Several regularization methods are known and used in quantum field theory: three and four dimensional momentum cutoff, Pauli-Villars type, dimensional regularization, lattice regularization, Schwinger's proper time method and others directly linked to renormalization like differential renormalization.
- Dimensional regularization (DREG) is the most popular and most appreciated as it respects the gauge and Lorentz symmetries of the Lagrangian. However DREG is not directly applicable to supersymmetric gauge theories. DREG gets rid of naive quadratic divergencies.
- In low energy effective field theories or in the Wilson renormalization group method there is an explicit cutoff, with well defined physical meaning. The cutoff gives the range of validity of the model.
- UV Regularization is an artificial algorithm that defines how to handle divergent momentum integrals. Here we show that with a small, reasonable modification sharp momentum cutoff can respect gauge (chiral and other) symmetries.
- Using naive momentum cutoff the QED vacuum polarization function will not be transverse.

$$\Pi_{\mu\nu}(q) = q_\mu q_\nu \Pi_L(q^2) - g_{\mu\nu} q^2 \Pi_T(q^2) \tag{1}$$

- The Ward identity reflect gauge invariance

$$q^\mu \Pi_{\mu\nu}(q) = 0 \tag{2}$$

- Symmetry preserving regularization must fulfill the Ward identities or equivalently freedom of momentum routing in loops.

Momentum cutoff via dimensional regularization

- DREG calculates in $d = 4 - 2\epsilon$, non-physical dimensions, singularities are $1/\epsilon$ poles.
- Originally quadratic divergencies set to zero, but Veltman realized those can be calculated in $d = 2 - 2(\epsilon - 1)$ in the limit $\epsilon \to 1$.
- Carefully calculating the one and two point Passarino-Veltman functions in DREG and 4-momentum cutoff the divergencies can be matched as [3, 4]

$$4\pi\mu^2\left(\frac{1}{\epsilon-1}+1\right) = \Lambda^2, \tag{3}$$

$$\frac{1}{\epsilon} - \gamma_E + \ln\left(4\pi\mu^2\right) + 1 = \ln\Lambda^2, \tag{4}$$

where μ is the mass-scale of dimensional regularization.

- Finite part of a divergent quantity is defined as

$$f_{\text{finite}} = \lim_{\epsilon\to 0}\left[f(\epsilon) - R(1)\left(\frac{1}{\epsilon-1}+1\right) - R(0)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + 1\right)\right], \tag{5}$$

where $R(1)$, $R(0)$ are the residues of the poles at $\epsilon = 1,\ 0$

- These identifications **define a unique symmetry preserving momentum cutoff** calculation based on the symmetry preserving DREG formulae.
- The recipe is to use the identification

$$k_\mu k_\nu \to \frac{1}{d} g_{\mu\nu} k^2, \tag{6}$$

here $d = 2$ for quadratic divergence, logaritmic divergence goes with $d = 4$ and a finite shift, and finite terms reproduce the standard $d = 4$.

Consistency conditions -gauge invariance

- Gauge theory calculations need a gauge invariant regulator.
- Simplest case a vacuum polarizarion function, later specified to QED

$$i\Pi_{\mu\nu}(q) = -(-ig)^2 \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_\mu \frac{\not k + m_a}{k^2 - m_a^2}\gamma_\nu \frac{\not q - \not k + m_b}{(q-k)^2 - m_b^2}\right). \tag{7}$$

Trace, Wick rotation ($k \to k_E$,), Feynman x-parameter, finite shift ($lE = k_E + xq_E$) of the loop momentum is performed. $\Delta = x(1-x)q_E^2 + (1-x)m_a^2 + xm_b^2$

- The Ward identity in QED for $m_a = m_b = m$

$$q^\mu \Pi_{\mu\nu}(q) = 0. \tag{8}$$

- Wihout evaluating the integrals and substituting anything for $k_\mu k_\nu$ get the constraint

$$\int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4}\frac{l_{E\mu} l_{E\nu}}{\left(l_E^2 + \Delta_1\right)^2} = \frac{1}{2} g_{\mu\nu}\int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4}\frac{1}{\left(l_E^2 + \Delta_1\right)}. \tag{9}$$

- Taylor expansion in q^2, x integration, $\Delta \to m$ get a condition for any m we can choose general *Delta*. We use the condition to **define the loop integral of** $l_{E\mu} l_{E\nu}$

$$\frac{l_{E\mu} l_{E\nu}}{\left(l_E^2 + \Delta\right)^{n+1}} \to \frac{1}{2n} g_{\mu\nu}\frac{1}{\left(l_E^2 + \Delta\right)^n} \tag{10}$$

- Relation with standard $k_\mu k_\nu \to \frac{1}{4} g_{\mu\nu} k^2$ substitution? Use

$$l_\mu l_\nu \to \frac{1}{d} g_{\mu\nu} l^2 \tag{11}$$

and d is different for various divergent terms. Integrate both sides of (9) and match the powers of Λ

$$\frac{1}{d}\Lambda^2 = \frac{1}{2}\Lambda^2, \tag{12}$$

$$\frac{1}{d}\left(\ln\left(\frac{\Lambda^2 + m^2}{m^2}\right) - \frac{1}{2}\right) = \frac{1}{4}\left(\ln\left(\frac{\Lambda^2 + m^2}{m^2}\right)\right), \tag{13}$$

$$\frac{1}{d} = \frac{1}{4} \text{ for finite terms.} \tag{14}$$

When the Wick-rotation is legal finite terms give $1/d = 1/4$. $d = 2$ for quadratic divergence and $d = 4$ and finite shift for log divergence. The identification partially found by [3, 4, 5].

Consistency conditions -momentum routing

- The choice of the internal momenta should not affect the result of loop calculations. A simple example is

$$\int d^4k \frac{k_\mu}{k^2 - m^2} - \int d^4k \frac{k_\mu + p_\mu}{(k+p)^2 - m^2} = 0 \tag{15}$$

expanding in powers of p at each order we get a condition (known from gauge invariance)

$$\int d^4k \left(\frac{p_\mu}{k^2 - m^2} - 2\frac{k_\mu k\cdot p}{k^2 - m^2}\right) = 0, \tag{16}$$

- Starting with $\int d^4k \frac{k_\mu}{(k^2-m^2)^n}$ it provides

$$p_\nu \int d^4k \left(\frac{g_{\mu\nu}}{\left(k^2 - m^2\right)^2} - 4\frac{k_\mu k_\nu}{\left(k^2 - m^2\right)^3}\right) = 0, \tag{17}$$

More indices in the nominator give further constraints.

Consistency conditions -vanishing surface terms

- All the conditions so far are related to a vanishing surface term, as

$$\int d^4k \frac{\partial}{\partial k^\nu}\left(\frac{k_\mu}{\left(k^2 + m^2\right)^n}\right) = \int d^4k \left(\frac{k_\mu k_\nu}{\left(k^2 + m^2\right)^{n+1}} - \frac{1}{2n} g_{\mu\nu}\frac{1}{\left(k^2 + m^2\right)^n}\right), \qquad n = 1, 2, \ldots \tag{18}$$

Surface terms vanish for finite integrals and symmetry respecting regularizations should give zero for divergent integrals also.

- Surface terms vanish in DREG. They vanish in any regularization where the previous consistency conditions are imposed, even in sharp momentum cutoff.

Improved momentum cutoff

- Simple sharp momentum cutoff is introduced to calculate the divergent integrals.
- Terms with definite Lorentz indices should calculated with the rule (10) or generalization for more indeces.
- One can insist on using the identification like $k_\mu k_\nu \to \frac{1}{d} g_{\mu\nu} k^2$, but then the rules (12,13, 14) apply to evaluate $1/d$. $d = 4$ is valid only for finite loop-integrals.
- In an effective composite Higgs model, the Fermion Condensate Model [8] oblique radiative corrections (S and T parameters) were calculated in DREG and with the improved cutoff, the finite results completely agree. The calculation involved vacuum polarization functions with two different fermion masses and no ambiguity appeared [10, 9].

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