A few recent developments in AdS/CFT with boundaries

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based on

- Z. Bajnok and L. Palla: JHEP 01 (2011) 011
- L. Palla: JHEP 03 (2011) 110

Plan

- AdS/CFT
- AdS/CFT with boundaries
- reflection matrices of bound states and the Yangian symmetry
- Lüscher type finite size corrections on the interval

  simplest AdS/CFT example

AdS/CFT

type IIB string in $AdS_5 \times S^5$ $\leftrightarrow$ $\mathcal{N} = 4$ $SU(N)$ Yang Mills in $1 + 3$

(Maldacena)

energy of a string state E $\leftrightarrow$ scaling dim. Δ of an operator in YM

global symmetries: bosonic $SO(4,2) \times SO(6) = SU(2,2) \times SU(4)$

isometry of $AdS_5 \times S^5$ conformal $+\mathcal{N} = 4$ R symmetry

+ SUSY

$\frac{SU(2,2|4)}{U(1)}$ $PSU(2,2|4)$ $\mathcal{N} = 4$ superconformal

$\lambda = g_{YM}^2 N$ $g_s = \frac{\lambda}{4\pi N}$ $\frac{R^2}{\alpha'} = \sqrt{\lambda}$ planar limit $N \to \infty$

single particle states with J large of the free string $\leftrightarrow$ long local gauge invariant single trace operators

integrability all λ-s are available $\updownarrow$ spin chain

$tr(ZZ\ldots Z)$ vacuum $PSU(2,2|4) \to PSU(2,2) \times PSU(2,2) \times R$

fundamental excitation magnon atypical short BPS representation 4d

$$E = \Delta - J = \sqrt{1 + 16 g^2 \sin^2\left(\frac{p}{2}\right)} \qquad \text{where} \qquad g = \sqrt{g_{YM}^2 N}/4\pi$$

integrability: YB + crossing magnon magnon S matrix known (Beisert, Arutyunov-Frolov-Zamaklar)

the centrally extended $su(2|2)$ algebra

$$\begin{aligned}
\left[\mathbb{L}_a{}^b, \mathbb{J}_c\right] &= \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c\,, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma\right] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma\,,\\
\left[\mathbb{L}_a{}^b, \mathbb{J}^c\right] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c\,, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma\right] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma\,,\\
\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\right\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}\,, \quad \left\{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\right\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger\,,\\
\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_b^{\dagger\beta}\right\} &= \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbb{H}\,,
\end{aligned}$$

$$a, b, \cdots \in \{1,2\} \qquad \alpha, \beta, \cdots \in \{3,4\}$$

for any Q there are Q magnon bound states (Chen-Dorey-Okamura, Arutyunov-Frolov) $4Q$ dim. atypical symmetric represetations superspace formalism: (Arutyunov-Frolov)

two bosonic (w_a) and two fermionic (θ_α) variables

$$\mathbb{L}_a^{\ b} = w_a \frac{\partial}{\partial w_b} - \frac{1}{2}\delta_a^b w_c \frac{\partial}{\partial w_c}\,, \qquad \mathbb{R}_\alpha^{\ \beta} = \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2}\delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma}\,,$$

$$\mathbb{Q}_\alpha^{\ a} = a\,\theta_\alpha \frac{\partial}{\partial w_a} + b\,\epsilon^{ab}\epsilon_{\alpha\beta} w_b \frac{\partial}{\partial \theta_\beta}\,, \qquad \mathbb{Q}_a^{\dagger\alpha} = d\,w_a \frac{\partial}{\partial \theta_\alpha} + c\,\epsilon_{ab}\epsilon^{\alpha\beta}\theta_\beta \frac{\partial}{\partial w_b}\,,$$

$$\mathbb{C} = ab\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\right), \qquad \mathbb{C}^\dagger = cd\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\right),$$

$$\mathbb{H} = (ad+bc)\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\right).$$

$$a = \sqrt{\frac{g}{2Q}}\eta \quad b = \sqrt{\frac{g}{2Q}}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right) \quad c = -\sqrt{\frac{g}{2Q}}\frac{\eta}{x^+} \quad d = \sqrt{\frac{g}{2Q}}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)$$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2Qi}{g} \qquad \frac{x^+}{x^-} = e^{ip} \quad \eta = e^{ip/4}\sqrt{i(x^- - x^+)}$$

Q magnon bound state $\mathcal{V}^Q(p)$, $\qquad 4Q = (Q+1) + (Q-1) + Q + Q$

$$Q+1 \to |j\rangle^1 = \frac{w_1^{Q-j} w_2^j}{\sqrt{(Q-j)!j!}} \qquad j = 0, \ldots Q$$

$$Q-1 \to |j\rangle^2 = \frac{w_1^{Q-2-j} w_2^j}{\sqrt{(Q-2-j)!j!}} \theta_3 \theta_4 \qquad j = 0, \ldots Q-2$$

$$Q \to |j\rangle^3 = \frac{w_1^{Q-1-j} w_2^j}{\sqrt{(Q-1-j)!j!}} \theta_3 \qquad j = 0, \ldots Q-1$$

$$Q \to |j\rangle^4 = \frac{w_1^{Q-1-j} w_2^j}{\sqrt{(Q-1-j)!j!}} \theta_4 \qquad j = 0, \ldots Q-1$$

for $\;Q = 1 \quad 4 = 2 + 1 + 1 \quad |0\rangle^1 = w_1, |1\rangle^1 = w_2, |0\rangle^3 = \theta_3, |0\rangle^4 = \theta_4$

attach open superstring to MGG $\longrightarrow$ $S^3 \subset S^5$ $S^5 : |W|^2 + |Y|^2 + |Z|^2 = 1$

(Hofman-Maldacena) integrability preserved when $\boxed{Y = 0}$ (or $Z = 0$)
gauge theory side: determinant type operators

$$\mathcal{O}_Y = \epsilon^{j_1 \dots j_{N-1} A}_{i_1 \dots i_{N-1} B} Y^{i_1}_{j_1} \dots Y^{i_{N-1}}_{j_{N-1}} (Z \dots Z \chi Z \dots Z \chi' Z \dots)^B_A$$

breaks $su(2,2)^2 \to su(2,1)^2$ no boundary degree of freedom

new object: reflection matrix $|0\rangle_B$ boundary vac. trivial vector sp. $\mathcal{V}(0)$

$\boxed{R(p) : \quad \mathcal{V}^Q(p) \otimes \mathcal{V}(0) \to \mathcal{V}^Q(-p) \otimes \mathcal{V}(0)}$ $\boxed{R(p) = \sum_i r_i(p) \Lambda_i}$

Λ_i invariant differential operators

integrability: BYB + boundary crossing unitarity $\longrightarrow$ $R(p)$

$su(2,1)$: $\mathbb{L}_1^{\ 1}$, $\mathbb{L}_2^{\ 2}$, $\mathbb{H}$, $\mathbb{R}_\alpha^{\ \beta}$, $\mathbb{Q}_\alpha^{\ 1}$, $\mathbb{Q}_1^{\dagger\alpha}$ symmetry $[\mathbb{J}^i, R]|j\rangle^a = 0$

for $Q=1$ symmetry determines $R(p)$ up to scalar

(Hofman-Maldacena, Ahn-Nepomechie)

scalar factor: boundary crossing unitarity (Hofman-Maldacena, Chen-Correa)

$$\mathbb{R}(p) = R_0(p)\text{diag}\left(-e^{i\frac{p}{2}}, e^{-i\frac{p}{2}}, 1, 1\right) \otimes \text{diag}\left(-e^{i\frac{p}{2}}, e^{-i\frac{p}{2}}, 1, 1\right)$$

$R_0(p) = -e^{-ip}\sigma(p,-p)$ $\quad$ $\sigma(p_1,p_2)$ dressing factor (BES)

for $Q = 2$ (Ahn-Nepomechie) symmetries not enough Yangian needed

(MacKay-Regelskis) description of the Yangian

for general Q (L P) Λ_i nondiagonal pieces $5Q-2$ unknown $A_l \dots E_l$

$$R = \sum_{l=0}^{Q} A_l \Lambda^l_{(1)} + \sum_{l=0}^{Q-2} B_l \Lambda^l_{(2)} + \sum_{l=0}^{Q-1} C_l \Lambda^l_{(3)} + \sum_{l=0}^{Q-2} D_l \Lambda^l_{(4)} + \sum_{l=0}^{Q-2} E_l \Lambda^l_{(5)}$$

$$\Lambda^l_{(1)} = \frac{w_1^{Q-l} w_2^l}{(Q-l)!l!} \frac{\partial^Q}{\partial w_1^{Q-l} \partial w_2^l}, \quad \dots \quad \Lambda^l_{(5)} = \frac{w_1^{Q-1-l} w_2^{l+1}}{(Q-2-l)!l!} \frac{\partial^{Q-2}}{\partial w_1^{Q-2-l} \partial w_2^l} \frac{\partial^2}{\partial \theta_4 \partial \theta_3}$$

Q of $A_l \dots E_l$ undetermined by symmetry

Yangian symmetry

Yangian extension $Y(\mathbf{g})$ of a *bulk* Lie symmetry $\mathbf{g}$: $\mathbb{J}^A$ grade 0 $\widehat{\mathbb{J}}^A$ grade 1

$[\mathbb{J}^A, \mathbb{J}^B] = f^{AB}_{C} \mathbb{J}^C$, $[\mathbb{J}^A, \widehat{\mathbb{J}}^B] = f^{AB}_{C} \widehat{\mathbb{J}}^C$ + Jacobi and Serre relations

$Y(su(2,2))$ known (Beisert)

evaluation representation $\widehat{\mathbb{J}}^A |u\rangle = -i\frac{g}{2} u \mathbb{J}^A |u\rangle$ (Beisert, AdLT)

(multi)magnons this form $u \equiv u(p) = \frac{1}{2}(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-})$

(MacKay et al.) boundary remnant $Y(\mathbf{h},\mathbf{g})$ if boundary $\mathbf{h} \subset \mathbf{g}$

$(\mathbf{h},\mathbf{g})$ sym. pair $\mathbf{g} = \mathbf{h} + \mathbf{m}$, $[\mathbf{h},\mathbf{h}] \subset \mathbf{h}$, $[\mathbf{h},\mathbf{m}] \subset \mathbf{m}$, $[\mathbf{m},\mathbf{m}] \subset \mathbf{h}$

$Y(\mathbf{h},\mathbf{g})$ generated by $(\mathbb{J}^i, \widetilde{\mathbb{J}}^p)$ $\widetilde{\mathbb{J}}^p = \widehat{\mathbb{J}}^p + \frac{1}{2} f^{p}_{qi} \mathbb{J}^q \mathbb{J}^i$ i, p $\mathbf{h}, \mathbf{m}$ indeces

for the $Y = 0$ brane $\mathbf{m}$ generated by $\mathbb{L}_2^{\ 1}$, $\mathbb{L}_1^{\ 2}$, $\mathbb{Q}_\gamma^{\ 2}$, $\mathbb{Q}_2^{\dagger\,\gamma}$, $\mathbb{C}$, $\mathbb{C}^\dagger$

$$\widetilde{\mathbb{Q}} \otimes 1 \equiv \triangle \widetilde{\mathbb{L}}_2^{\ 1} = \left(\widehat{\mathbb{L}}_2^{\ 1} + \frac{1}{2} \left(\mathbb{L}_2^{\ 1} \mathbb{L}_1^{\ 1} - \mathbb{L}_2^{\ 1} \mathbb{L}_2^{\ 2} - \mathbb{Q}_2^{\dagger\gamma} \mathbb{Q}_\gamma^{\ 1} \right) \right) \otimes 1$$

R $\widetilde{\mathbb{Q}}$ diagonal on $|j\rangle^\alpha$ requiring $[\widetilde{\mathbb{Q}}, R] |j\rangle^\gamma = 0$ determines C_j

$$C_{j+1} = \Phi(j) C_j, \quad j = 0, \ldots, Q-2, \qquad \text{where} \quad \Phi(j) = \frac{i\frac{g}{2}u + \frac{Q}{2} - j - 1}{-i\frac{g}{2}u + \frac{Q}{2} - j - 1}$$

normalization $\quad A_0 = 1 \quad C_0 = A_0 \frac{d}{\dot{d}} = e^{-ip/2}$

$$A_Q = \frac{c}{\dot{c}} C_{Q-1} = e^{-ip} \prod_{l=0}^{Q-2} \Phi(l) = (-1)^Q e^{-ip}$$

$$\begin{aligned} A_{j+1} &= \left(\prod_{l=0}^{j-1} \Phi(l)\right) \frac{(Q-1-j)\Phi(j) x^+ - (j+1)/x^+}{(Q-1-j)x^+ + (j+1)/x^-} \\ B_j &= \left(\prod_{l=0}^{j-1} \Phi(l)\right) \frac{(Q-1-j)x^- - (j+1)\Phi(j)/x^-}{(Q-1-j)x^+ + (j+1)/x^-} \\ E_j &= -D_j = e^{-ip/2} \left(\prod_{l=0}^{j-1} \Phi(l)\right) \frac{x^+ \Phi(j) + x^-}{x^+ x^- (Q-1-j) + (j+1)} \end{aligned}$$

$j = 0, \ldots Q-2$

scalar factor fusion method (Ahn-Bak-Rey)

Boundary finite size corrections for multiparticle states

$\{\Delta$ of single trace with $J = \#$ of fields $\to \infty\}$ $\to$ asymptotic Bethe Ansatz (ABA) all $1/J$ corrections

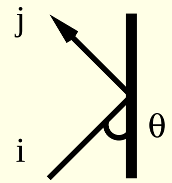
large but finite J: wrapping effects string side: vacuum polarization
Lüscher corrections: ∞ data $\to$ exponentially small finite size corr. $\sim e^{-J}$

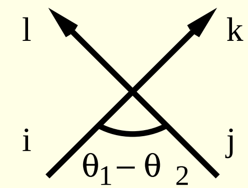
bulk: 4 and 5 loop Konishi $\to$ exact gauge th. computation (Bajnok-Janik)

next: finite size corrections for determinant type operators / open strings
groundstates of $Y = 0$ and $Z = 0$ branes (Correa-Young)
excited states (multiparticle) (Bajnok-Palla)

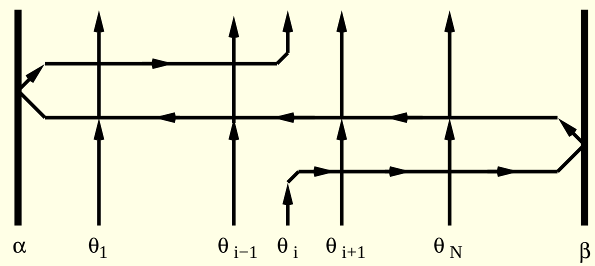
ABA $\longrightarrow$ boundary Bethe-Yang eq. polynomial corrections
boundary Lüscher corrections not known even for relativistic case

relativistic case first $E = m\cosh\theta \quad p = m\sinh\theta \quad \theta$ rapidity

$\mathbb{R}(\theta) = R_i^j(\theta)$ 

$\mathbb{S}(\theta_1-\theta_2) = S_{ij}^{kl}(\theta_1-\theta_2)$ 

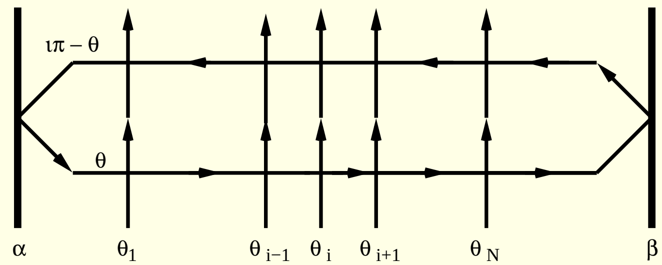
boundary Bethe Yang eq.



$E = \sum_{i=1}^{N} E(\theta_i)$

$$e^{2ip(\theta_i)L}\prod_{j=i+1}^{N}\mathbb{S}(\theta_i-\theta_j)\mathbb{R}_\beta(\theta_i)\prod_{j=N}^{1}\mathbb{S}(\theta_j+\theta_i)\mathbb{R}_\alpha(\theta_i)\prod_{j=1}^{i-1}\mathbb{S}(\theta_i-\theta_j) = \mathbb{I} \quad \theta_i > 0$$

can be derived from double row transfer matrix (DTM) $\mathbb{T}$



$$\mathbb{R}^c = \mathbb{C}\mathbb{R}\mathbb{C}^{-1}$$

$$\mathbb{T}(\theta|\theta_1,\dots,\theta_N) = \mathrm{Tr}\left(\prod_{j=1}^{N}\mathbb{S}(\theta-\theta_j)\mathbb{R}_\beta(\theta)\prod_{j=N}^{1}\mathbb{S}(\theta_j+\theta)\mathbb{R}^c_\alpha(i\pi-\theta)\right)$$

YB and BYB guarantee $[\mathbb{T}(\theta|\theta_1,\dots,\theta_N),\mathbb{T}(\lambda|\theta_1,\dots,\theta_N)] = 0$

eigenvalue $t(\theta|\theta_1,\dots,\theta_N)$ $\boxed{Y_{as}(\theta|\theta_1,\dots,\theta_N) = e^{2ip(\theta)L}\,t(\theta|\theta_1,\dots,\theta_N)}$

BBY: $Y_{as}(\theta_i|\theta_1,\dots,\theta_N) = -1 \qquad i = 1,\dots N$

Lüscher correction (vacuum polarization) of N particle energy

$$\Delta E = -\int_0^\infty \frac{d\theta}{2\pi}\partial_\theta p(\theta)\,Y_{as}(\theta+i\frac{\pi}{2}|\theta_1,\dots,\theta_N)$$

derived for diagonal reflections / scattering (from BTBA) (boundary Lie-Yang)

for ground state

checked for Dirichlet sine-Gordon (NLIE) accept for non relativistic models $\mathbb{S}(u_i, u_j)$ u_i generalized rapidity

unitarity: $\mathbb{S}(u_1, u_2) = \mathbb{S}(u_2, u_1)^{-1}$

crossing: $\mathbb{S}^{c_1}(u_1, u_2) = \mathbb{S}(u_2, u_1 - \omega)$ $\mathbb{R}(u) = \mathbb{S}(u, -u)\mathbb{R}^c(\omega - u)$ crossing parameter ω

$\mathbb{T}(u|u_1, \ldots, u_N)$ $Y_{as}(u|u_1, \ldots, u_N)$ formally the same

BBY equations $Y_{as}(u_i|u_1, \ldots, u_N) = -1$

N particle energy correction

$$\Delta E = -\int_0^\infty \frac{du}{2\pi} \partial_u \tilde{p}(u)\, Y_{as}(u + \frac{\omega}{2}|u_1, \ldots, u_N)$$

u continued into ‘mirror’ domain $u \to u + \frac{\omega}{2}$

mirror theory: double Wick rotation

$$\tilde{E}(u) = -ip(u + \frac{\omega}{2}) \qquad \tilde{p}(u) = -iE(u + \frac{\omega}{2})$$

non relativistic case: mirror $\neq$ original simplest Lüscher correction in AdS/CFT (for the $Y=0$ boundary)

$E^2 - 16g^2\sin^2\frac{p}{2} = Q^2$ on torus z (generalized rapidity) $\omega=\omega_2$

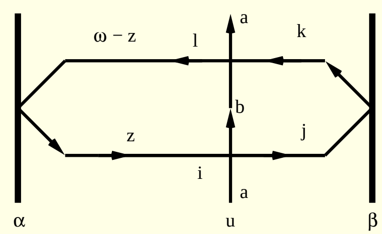
$p = 2\,\mathrm{am}(z,k)$ $E = Q\,\mathrm{dn}(z,k)$ $k=-16\frac{g^2}{Q^2}$ $2\omega_2 = 4iK(1-k)-4K(k)$

checked vacuum's vanishing correction (Correa-Young) reproduced

one particle BBY on a strip of width L

$e^{-2ip(L+1)}\sigma(p,-p)^2\mathrm{diag}(e^{ip},e^{-ip},1,1)\otimes\mathrm{diag}(e^{ip},e^{-ip},1,1)=1$

for a $(1,1)$ magnon $p_n = n\frac{\pi}{L}$ shortest strip $L=2$ $n=1$



$\Delta E_a(L) = -\sum_Q \int_0^{\frac{\omega_1}{2}} \frac{dz}{2\pi}(\partial_z\tilde{p}_Q(z))\mathbb{S}_{ia}^{jb}(\frac{\omega}{2}+z,u)\mathbb{R}_j^k(\frac{\omega}{2}+z)\mathbb{S}_{lb}^{ka}(\frac{\omega}{2}-z,u)\mathbb{C}^{l\bar{l}}\mathbb{R}_{\bar{l}}^{\bar{i}}(\frac{\omega}{2}-z)\mathbb{C}_{\bar{i}i}e^{-2\tilde{\epsilon}_Q L}$ infinite sum over the mirror boundstates

simplest Lüscher correction in AdS/CFT (for the $Y=0$ boundary)

$E^2 - 16g^2\sin^2\frac{p}{2} = Q^2$ on torus z (generalized rapidity) $\boxed{\omega=\omega_2}$

$p = 2\,\mathrm{am}(z,k)$ $E = Q\,\mathrm{dn}(z,k)$ $k = -16\frac{g^2}{Q^2}$ $2\omega_2 = 4iK(1-k) - 4K(k)$

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$e^{-2ip(L+1)}\sigma(p,-p)^2\mathrm{diag}(e^{ip},e^{-ip},1,1)\otimes\mathrm{diag}(e^{ip},e^{-ip},1,1)=1$

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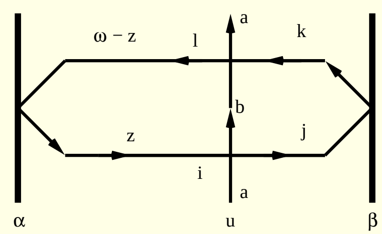

$\Delta E_a(L) = -\sum_Q \int_0^{\frac{\omega_1}{2}} \frac{dz}{2\pi}(\partial_z \tilde{p}_Q(z))\mathbb{S}_{ia}^{jb}(\frac{\omega}{2}+z,u)\mathbb{R}_j^k(\frac{\omega}{2}+z)\mathbb{S}_{lb}^{ka}(\frac{\omega}{2}-z,u)\mathbb{C}^{l\bar{l}}\mathbb{R}_{\bar{l}}^{\bar{i}}(\frac{\omega}{2}-z)\mathbb{C}_{\bar{i}i}e^{-2\tilde{\epsilon}_Q L}$ infinite sum over the mirror boundstates

$$\Delta E = -768g^{16}(80\zeta(7)-220\zeta(11)+143\zeta(13))$$