Y system for $Y = 0$ brane in planar AdS/CFT

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based on

- Z. Bajnok and L. Palla: JHEP 01 (2011) 011
- Z.Bajnok, R.I. Nepomechie, L. Palla and R. Suzuki: arXiv:1205.2060 [hep-th]

Plan

- AdS/CFT
- AdS/CFT with boundaries
- Lüscher type finite size corrections on the interval

  simplest AdS/CFT example
- Y system and asymptotical solutions
- checking the Y system
- ABA, generating functional, duality

AdS/CFT

type IIB string in $AdS_5 \times S^5$ $\leftrightarrow$ $\mathcal{N} = 4$ $SU(N)$ Yang Mills in $1 + 3$

(Maldacena)

energy of a string state E $\leftrightarrow$ scaling dim. Δ of an operator in YM

global symmetries: bosonic $SO(4,2) \times SO(6) = SU(2,2) \times SU(4)$

isometry of $AdS_5 \times S^5$ conformal $+\mathcal{N} = 4$ R symmetry

+ SUSY

$\frac{SU(2,2|4)}{U(1)}$ $PSU(2,2|4)$ $\mathcal{N} = 4$ superconformal

$\lambda = g_{YM}^2 N$ $g_s = \frac{\lambda}{4\pi N}$ $\frac{R^2}{\alpha'} = \sqrt{\lambda}$ planar limit $N \to \infty$

single particle states with J large $\leftrightarrow$ long local gauge invariant

of the free string single trace operators

integrability all λ-s are available $\updownarrow$

spin chain

$tr(ZZ\ldots Z)$ vacuum $PSU(2,2|4) \to PSU(2,2) \times PSU(2,2) \times R$

fundamental excitation magnon atypical short BPS representation 4d

$$E = \Delta - J = \sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)} \quad \text{where} \quad g = \sqrt{g_{YM}^2 N}/4\pi$$

integrability: YB + crossing magnon magnon S matrix known (Beisert, Arutyunov-Frolov-Zamaklar)

for any Q there are Q magnon bound states (Chen-Dorey-Okamura, Arutyunov-Frolov) $4Q$ dim. atypical symmetric representations $\mathcal{V}^Q(p)$

rapidity parameter u

$$E_Q(u) = Q + 2ig\left(\frac{1}{x^{[Q]}} - \frac{1}{x^{[-Q]}}\right) \qquad p_Q(u) = -i \log \frac{x^{[Q]}}{x^{[-Q]}}$$

$$x(u) = \frac{u}{2g} + \sqrt{\frac{u}{2g} - 1}\sqrt{\frac{u}{2g} + 1} \qquad f^{[n]}(u) = f(u + \frac{in}{2})$$

mirror model by double Wick rotation $p \to -i\tilde{\epsilon} \quad E \to -i\tilde{p}$

$$x(u) = \frac{u}{2g} + i\sqrt{1 - \frac{u^2}{4g^2}}$$

AdS/CFT with boundaries

attach open superstring to MGG $\longrightarrow$ $S^3 \subset S^5$ $S^5 : |W|^2 + |Y|^2 + |Z|^2 = 1$

(Hofman-Maldacena) integrability preserved when $\boxed{Y = 0}$ (or $Z = 0$)
gauge theory side: determinant type operators

$$\mathcal{O}_Y = \epsilon^{j_1 \dots j_{N-1} A}_{i_1 \dots i_{N-1} B} Y^{i_1}_{j_1} \dots Y^{i_{N-1}}_{j_{N-1}} (Z \dots Z \chi Z \dots Z \chi' Z \dots)^B_A$$

breaks $su(2,2)^2 \to su(2,1)^2$ no boundary degree of freedom

new object: reflection matrix $|0\rangle_B$ boundary vac. trivial vector sp. $\mathcal{V}(0)$
$\boxed{R(p): \quad \mathcal{V}^Q(p) \otimes \mathcal{V}(0) \to \mathcal{V}^Q(-p) \otimes \mathcal{V}(0)}$ $\boxed{R(p) = \sum_i r_i(p) \Lambda_i}$
Λ_i invariant differential operators
integrability: BYB + boundary crossing unitarity $\longrightarrow$ $R(p)$
$su(2,1)$: symmetry $[\mathbb{J}^i, R] |j\rangle^a = 0$

for $Q=1$ symmetry determines $R(p)$ up to scalar

(Hofman-Maldacena, Ahn-Nepomechie)

scalar factor: boundary crossing unitarity (Hofman-Maldacena, Chen-Correa)

$$\mathbb{R}(p) = R_0(p)\text{diag}\left(e^{-i\frac{p}{2}}, -e^{i\frac{p}{2}}, 1, 1\right) \otimes \text{diag}\left(e^{-i\frac{p}{2}}, -e^{i\frac{p}{2}}, 1, 1\right)$$

$R_0(p) = -e^{-ip}\sigma(p,-p)$ $\sigma(p_1,p_2)$ dressing factor (BES)

for $Q = 2$ (Ahn-Nepomechie) symmetries not enough Yangian needed

(MacKay-Regelskis) description of the Yangian

for general Q (L P) Λ_i nondiagonal pieces $5Q-2$ unknown

Yangian determines R up to scalar

Boundary finite size corrections for multiparticle states

$\{\Delta$ of single trace with $J = \#$ of fields $\to \infty\}$ $\to$ asymptotic Bethe Ansatz (ABA) all $1/J$ corrections

large but finite J: wrapping effects string side: vacuum polarization
Lüscher corrections: ∞ data $\to$ exponentially small finite size corr. $\sim e^{-J}$

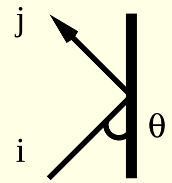
bulk: 4 and 5 loop Konishi $\to$ exact gauge th. computation (Bajnok-Janik)

next: finite size corrections for determinant type operators / open strings
groundstates of $Y = 0$ and $Z = 0$ branes (Correa-Young)
excited states (multiparticle) (Bajnok-Palla)

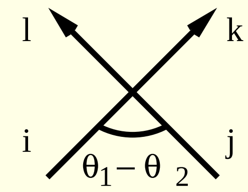
ABA $\longrightarrow$ boundary Bethe-Yang eq. polynomial corrections
boundary Lüscher corrections not known even for relativistic case

relativistic case first $E = m\cosh\theta \quad p = m\sinh\theta \quad \theta$ rapidity

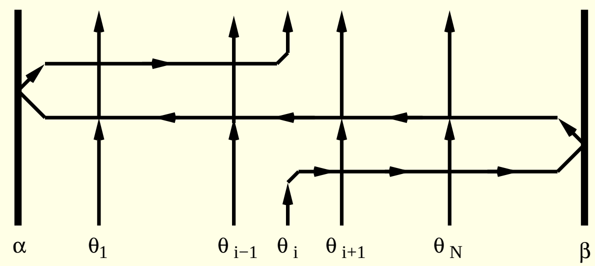
$\mathbb{R}(\theta) = R_i^j(\theta)$



$\mathbb{S}(\theta_1 - \theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)$



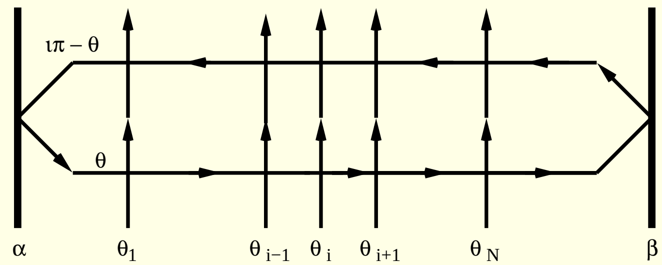
boundary Bethe Yang eq.



$E = \sum_{i=1}^{N} E(\theta_i)$

$$e^{2ip(\theta_i)L} \prod_{j=i+1}^{N} \mathbb{S}(\theta_i - \theta_j)\mathbb{R}_\beta(\theta_i) \prod_{j=N}^{1} \mathbb{S}(\theta_j + \theta_i)\mathbb{R}_\alpha(\theta_i) \prod_{j=1}^{i-1} \mathbb{S}(\theta_i - \theta_j) = \mathbb{I} \quad \theta_i > 0$$

can be derived from double row transfer matrix (DTM) $\mathbb{T}$



$$\mathbb{R}^c = \mathbb{C}\mathbb{R}\mathbb{C}^{-1}$$

$$\mathbb{T}(\theta|\theta_1,\dots,\theta_N) = \mathrm{Tr}\left(\prod_{j=1}^{N}\mathbb{S}(\theta-\theta_j)\mathbb{R}_\beta(\theta)\prod_{j=N}^{1}\mathbb{S}(\theta_j+\theta)\mathbb{R}_\alpha^c(i\pi-\theta)\right)$$

YB and BYB guarantee $[\mathbb{T}(\theta|\theta_1,\dots,\theta_N),\mathbb{T}(\lambda|\theta_1,\dots,\theta_N)] = 0$

eigenvalue $t(\theta|\theta_1,\dots,\theta_N)$ $\boxed{Y_{as}(\theta|\theta_1,\dots,\theta_N) = e^{2ip(\theta)L}\,t(\theta|\theta_1,\dots,\theta_N)}$

BBY: $Y_{as}(\theta_i|\theta_1,\dots,\theta_N) = -1 \qquad i = 1,\dots N$

Lüscher correction (vacuum polarization) of N particle energy

$$\Delta E = -\int_0^\infty \frac{d\theta}{2\pi}\partial_\theta p(\theta)\, Y_{as}(\theta + i\frac{\pi}{2}|\theta_1,\dots,\theta_N)$$

derived for diagonal reflections / scattering (from BTBA) (boundary Lie-Yang)

for ground state

checked for Dirichlet sine-Gordon (NLIE) accept for non relativistic models $\mathbb{S}(u_i, u_j)$ u_i rapidity

unitarity: $\mathbb{S}(u_1, u_2) = \mathbb{S}(u_2, u_1)^{-1}$

crossing: $\mathbb{S}^{c_1}(u_1, u_2) = \mathbb{S}(u_2, u_1 - \omega)$ $\mathbb{R}(u) = \mathbb{S}(u, -u)\mathbb{R}^c(\omega - u)$ crossing parameter ω

$\mathbb{T}(u|u_1, \ldots, u_N)$ $Y_{as}(u|u_1, \ldots, u_N)$ formally the same

BBY equations $Y_{as}(u_i|u_1, \ldots, u_N) = -1$

N particle energy correction

$$\Delta E = -\int_0^\infty \frac{du}{2\pi} \partial_u \tilde{p}(u)\, Y_{as}(u + \frac{\omega}{2}|u_1, \ldots, u_N)$$

u continued into 'mirror' domain $u \to u + \frac{\omega}{2}$

mirror theory: double Wick rotation

$$\tilde{E}(u) = -ip(u + \frac{\omega}{2}) \qquad \tilde{p}(u) = -iE(u + \frac{\omega}{2})$$

non relativistic case: mirror $\neq$ original simplest Lüscher correction in AdS/CFT (for the $Y=0$ boundary)

$E^2 - 16g^2 \sin^2 \frac{p}{2} = Q^2$ on torus z (generalized rapidity) $\boxed{\omega = \omega_2}$

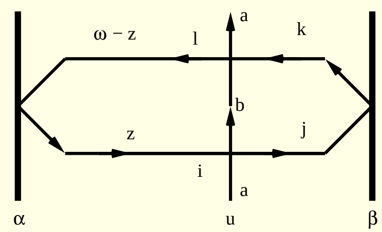
$p = 2\,\mathrm{am}(z,k)$ $E = Q\,\mathrm{dn}(z,k)$ $k = -16\frac{g^2}{Q^2}$ $2\omega_2 = 4iK(1-k) - 4K(k)$

checked vacuum's vanishing correction (Correa-Young) reproduced

one particle BBY on a strip of width L

$$\boxed{e^{-2ip(L+1)}\sigma(p,-p)^2 \mathrm{diag}(e^{-ip}, e^{ip}, 1, 1) \otimes \mathrm{diag}(e^{-ip}, e^{ip}, 1, 1) = 1}$$

for a $(2,\dot{2})$ magnon $p_n = n\frac{\pi}{L}$ shortest strip $L=2$ $n=1$



$\Delta E_a(L) = -\sum_Q \int_0^{\frac{\omega_1}{2}} \frac{dz}{2\pi} (\partial_z \tilde{p}_Q(z)) \mathbb{S}_{ia}^{jb}(\frac{\omega}{2}+z,u) \mathbb{R}_j^k(\frac{\omega}{2}+z) \mathbb{S}_{lb}^{ka}(\frac{\omega}{2}-z,u) \mathbb{C}^{l\bar{l}} \mathbb{R}_{\bar{l}}^{\bar{i}}(\frac{\omega}{2}-z) \mathbb{C}_{\bar{i}i} e^{-2\tilde{\epsilon}_Q L}$ infinite sum over the mirror boundstates

simplest Lüscher correction in AdS/CFT (for the $Y=0$ boundary)

$E^2-16g^2\sin^2\frac{p}{2}=Q^2$ on torus z (generalized rapidity) $\boxed{\omega=\omega_2}$

$p=2\,\mathrm{am}(z,k)$ $E=Q\,\mathrm{dn}(z,k)$ $k=-16\frac{g^2}{Q^2}$ $2\omega_2=4iK(1-k)-4K(k)$

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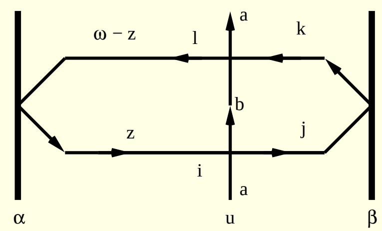
one particle BBY on a strip of width L

$$\boxed{e^{-2ip(L+1)}\sigma(p,-p)^2\mathrm{diag}(e^{-ip},e^{ip},1,1)\otimes\mathrm{diag}(e^{-ip},e^{ip},1,1)=1}$$

for a $(2,\dot{2})$ magnon $p_n=n\frac{\pi}{L}$ shortest strip $L=2$ $n=1$



$$\Delta E_a(L)=-\sum_Q\int_0^{\frac{\omega_1}{2}}\frac{dz}{2\pi}(\partial_z\tilde{p}_Q(z))\mathbb{S}_{ia}^{jb}(\tfrac{\omega}{2}+z,u)\mathbb{R}_j^k(\tfrac{\omega}{2}+z)\mathbb{S}_{lb}^{ka}(\tfrac{\omega}{2}-z,u)\mathbb{C}^{l\bar{l}}\mathbb{R}_{\bar{l}}^{\bar{i}}(\tfrac{\omega}{2}-z)\mathbb{C}_{\bar{i}i}e^{-2\tilde{\epsilon}_QL}$$

infinite sum over the mirror boundstates

$$\Delta E_{2\dot{2}}(\frac{\pi}{2})=192g^{12}(4\zeta(5)-7\zeta(9))$$

Y system and asymptotical solution

Y system $+$ asymptotical $+$ analytical info $\rightarrow$ unique sol. of spectral problem in planar AdS/CFT (closed strings) (Gromov Kazakov Vieira, Kuniba Nakanishi Suzuki)

extend Y system to AdS/CFT with boundaries (open strings)

idea Y system same as for closed strings
asymptotical and analytical properties different

examples periodic β deformed AdS/CFT (Gromov Leskovich-Maslyuk, Ahn Bajnok Bombardelli Nepomechie)
b.c. of Wilson loop $\rightarrow$ (B)TBA $\rightarrow$ Y system (Correa Maldacena Server, Drukker)

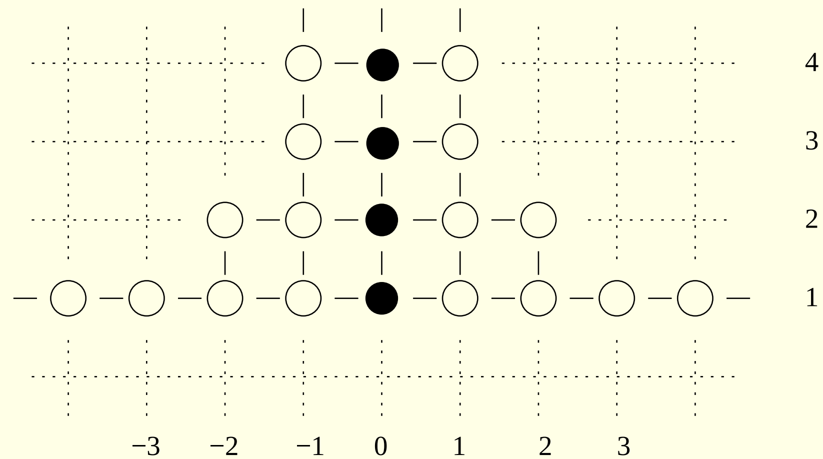
closed strings/periodic Y functions $\leftrightarrow$ (eigenvals of) transfer matrices
open strings/boundary Y functions $\leftrightarrow$ " double row transfer matrices

Y system $PSU(2,2|4)$ symmetry $Y_{a,s}(u)$ Y functions

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a-1,s}Y_{a+1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a-1,s})(1+Y_{a+1,s})}$$

$$f^{\pm}(u) = f(u \pm \tfrac{i}{2}) \quad Y_{0,s} = \infty$$

$$Y_{2,|s|>2} = \infty \quad Y_{a>2,\pm 2} = 0$$



energy of fundamental multi-particle state with p_k in terms of $Y_Q = Y_{Q,0}$

$$E(L) = \sum_k E_1(p_k) - \sum_{Q=1}^{\infty} \int \frac{du}{2\pi} \partial_u \tilde{p}_Q \log(1 + Y_Q)$$

momenta by exact Bethe eq. $Y_1(p_k) = -1$

set up for both periodic and boundary difference: asymptotic behaviour

Asymptotic solutions of the Y system

T system $\quad T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1} \qquad Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$

$L \to \infty \quad Y_{a,0} \to 0 \quad PSU(2,2|4)$ T system splits $SU(2|2)$ ones $\quad T_{a,0} = 1$

leading order $\quad Y_{a,0} = \frac{\phi^{[-a]}}{\phi^{[a]}}T_{a,-1}T_{a,1} \qquad$ fixed from Lüscher correction

periodic case (Bajnok Janik) $\qquad Y_{a,0} = e^{-\tilde{\epsilon}_a L}\mathbb{T}_a$

$$\mathbb{T}_a(p,\{p_i\}) = \mathrm{sTr}_a(\mathbb{S}_{aN}(p,p_N)\dots\mathbb{S}_{a1}(p,p_1))$$

S matrix factorizes $\quad \mathbb{S} = S_0\, S \otimes \dot{S} \quad \to \quad \mathbb{T}_a = t_a\, t_{a,1} \otimes \dot{t}_{a,1}$

$$t_{a,1}(p\,;\{p_i\}) = \mathrm{sTr}(S_{aN}(p,p_N)\cdots S_{a1}(p,p_1))$$

$t_a = t_1^{[1-a]}t_1^{[3-a]}\dots t_1^{[a-3]}t_1^{[a-1]} \qquad t_1 = \prod_{i=1}^N S_0(p,p_i)$

$SU(2|2)$ T functions $\quad$ left/right $SU(2|2)$ transfer matrices $\quad \frac{\phi^-}{\phi^+} = \left(\frac{x^-}{x^+}\right)^L t_1$

boundary case (Bajnok Palla, BNPS) $\quad Y_{a,0} = e^{-2\tilde{\epsilon}_a L}\mathbb{D}_a$

$$\mathbb{D}_a(p,\{p_i\}) = \mathrm{Tr}_a(\mathbb{S}_{aN}(p,p_N)\dots\mathbb{S}_{a1}(p,p_1)\mathbb{R}_a^-(p)\times$$
$$\mathbb{S}_{1a}(p_1,-p)\dots\mathbb{S}_{Na}(p_N,-p)\tilde{\mathbb{R}}_a^+(-p))$$

$\mathbb{R}_a^-(p)_\gamma^\beta = \mathbb{S}_{aa}(p,-p)_{\alpha\gamma}^{\beta\delta}\tilde{\mathbb{R}}_a^+(-p)_\delta^\alpha$ ensures $Y_{1,0}(p_i) = -1$ equivalent to the boundary Bethe-Yang equations

reflection matrices factorize $\mathbb{R}^- = R_0^- R^- \otimes \dot{R}^- \qquad \tilde{\mathbb{R}}^+ = R_0^+ \tilde{R}^+ \otimes \dot{\tilde{R}}^+$

double row transfer matrix also $\mathbb{D}_a = d_a\, d_{a,1} \otimes \dot{d}_{a,1}$

$$d_a = d_1^{[1-a]} d_1^{[3-a]} \dots d_1^{[a-3]} d_1^{[a-1]} \qquad d_1 = R_0^-(p) R_0^+(-p) \prod_{i=1}^{N} S_0(p,p_i) S_0(p_i,-p)$$

asymptotic solution of T system

$$T_{a,1} = d_{a,1} \qquad T_{a,-1} = \dot{d}_{a,1} \qquad \frac{\phi^-}{\phi^+} = e^{-2\tilde{\epsilon}_1 L} d_1$$

calculation of $d_{a,1}$ generating functional (Kazakov Sorin Zabrodin, Gromov Kazakov)

Some notation

$x(u) + \frac{1}{x(u)} = \frac{u}{g}$ branch points at $u = \pm 2g$

for N-particle ground state $|1, 1, \ldots, 1\rangle$ we use

$$R^{(\pm)} = \prod_{i=1}^{N} \left(x(p) - x^{\mp}(p_i)\right)\left(x(p) + x^{\pm}(p_i)\right) \quad Q(u) = \prod_{i=1}^{N}(u - u_i)(u + u_i)$$

$$B^{(\pm)} = \prod_{i=1}^{N} \left(\frac{1}{x(p)} - x^{\mp}(p_i)\right)\left(\frac{1}{x(p)} + x^{\pm}(p_i)\right)$$

for generic state (m_1 y roots m_2 w roots)

$$B_1 R_3 = \prod_{j=1}^{m_1} \left(x(p) - y_j\right)\left(x(p) + y_j\right) \quad R_1 B_3 = \prod_{j=1}^{m_1} \left(\frac{1}{x(p)} - y_j\right)\left(\frac{1}{x(p)} + y_j\right)$$

$$Q_1(u) = \prod_{j=1}^{m_1} \left(\frac{u}{g} - y_j - \frac{1}{y_j}\right)\left(\frac{u}{g} + y_j + \frac{1}{y_j}\right) = \left(\prod_{j=1}^{m_1} -\frac{1}{y_j^2}\right) B_1 R_3 R_1 B_3$$

$$Q_2(u) = \prod_{l=1}^{m_2}(u - w_l)(u + w_l)$$

Generating functional

$SU(2)$ sector first $\quad Q=1$ particles $\quad S^{11}_{11}(x_1,x_2)=1$

$$\tilde{d}_{1,1} = \mathrm{sTr}_1\left(S_{1N}(p,p_N)\dots S_{11}(p,p_1)R_1^-(p)S_{11}(p_1,-p)\dots S_{N1}(p_N,-p)R_1^-(-p)\right)$$

$R^-(-p) \propto (-1)^F \tilde{R}^+(-p)$ changes trace to supertrace

$$\mathbb{D}_1(p) = \tilde{d}_1(p)\tilde{d}_{1,1}(p)\otimes\dot{\tilde{d}}_{1,1}(p) \qquad \tilde{d}_1(p) \qquad \text{later}$$

$|1,1,\dots,1\rangle$ ground state eigenvalue of $\tilde{d}_{1,1}(p)$

$$\boxed{\Lambda^{su(2)}(p) = \rho_1\Lambda_1 + \rho_2\Lambda_2 - \rho_3\Lambda_3 - \rho_4\Lambda_4}$$

$$\Lambda_1 = 1 \qquad \Lambda_2 = \frac{R^{(-)+}}{R^{(+)+}}\frac{B^{(-)-}}{B^{(+)-}} \qquad \Lambda_3 = \Lambda_4 = \frac{R^{(-)+}}{R^{(+)+}}$$

$$\rho_1 = \frac{(1+(x^-)^2)(x^-+x^+)}{2x^+(1+x^+x^-)} \qquad \rho_2 = \frac{x^-(x^-+x^+)(1+(x^+)^2)}{2(x^+)^2(1+x^-x^+)}$$

$$\rho_3+\rho_4 = \frac{(x^-+x^+)^2}{2(x^+)^2}$$

analogy with periodic case $\mathcal{D} = e^{-\frac{i}{2}\partial_u}$

$$\begin{aligned}\tilde{\mathcal{W}}^{-1} &= (1-\mathcal{D}\rho_1\Lambda_1\mathcal{D})(1-\mathcal{D}\rho_3\Lambda_3\mathcal{D})^{-1}(1-\mathcal{D}\rho_4\Lambda_4\mathcal{D})^{-1}(1-\mathcal{D}\rho_2\Lambda_2\mathcal{D})\\ &= \sum_a (-1)^a\mathcal{D}^a\tilde{d}_{a,1}\mathcal{D}^a\end{aligned}$$

no particle $\tilde{\mathcal{W}} = 1 \longrightarrow \rho_1 = \rho_3 \quad \rho_2 = \rho_4 \quad \frac{\rho_2}{\rho_1} = \frac{\rho_4}{\rho_3} = \frac{u^+}{u^-}$

change normalization

$$\Lambda^{su(2)}(p) = \rho_3\Lambda_3\left(\frac{\rho_1}{\rho_3}\Lambda_3^{-1} + \frac{\rho_2}{\rho_3}\frac{\Lambda_2}{\Lambda_3} - 1 - \frac{\rho_4}{\rho_3}\right) = \rho_3\Lambda_3\hat{\Lambda}^{su(2)}$$

using $\hat{\Lambda}$ $\mathcal{W}^{-1}_{su(2)} = \sum_a (-1)^a\mathcal{D}^a\hat{d}_{a,1}\mathcal{D}^a$

$$\begin{aligned}(-1)^a\hat{d}_{a,1} &= (a+1)\frac{u}{u^{[-a]}} - a\frac{u^-}{u^{[-a]}}\frac{R^{(+)[a]}}{R^{(-)[a]}} - a\frac{u^+}{u^{[-a]}}\frac{B^{(-)[-a]}}{B^{(+)[-a]}}\\ &+ (a-1)\frac{u}{u^{[-a]}}\frac{R^{(+)[a]}}{R^{(-)[a]}}\frac{B^{(-)[-a]}}{B^{(+)[-a]}}\end{aligned}$$

$\hat{d}_{2,1}$ tested against eigenvalues of $d_{2,1}$ for $N = 1, 2, 3$

also $\mathcal{W}_{su(2)} = \sum_s \mathcal{D}^s\hat{d}_{1,s}\mathcal{D}^s$ constructed $\hat{d}_{1,2}$ tested similarly

Checking the Y functions

normalization BBY $(1\dot{1})$ particle $p_{1\dot{1}} = \frac{\pi}{L+2}n$ $(2\dot{2})$ particle $p_{2\dot{2}} = \frac{\pi}{L}n$

recover BBY from $Y_1(p_n) = -1 \quad \longleftrightarrow \quad \mathbb{D}_1(p_1|p_1)e^{-2ip_1L} = -1$

$\tilde{d}_1(p) = S_0(p,p_1)S_0(p_1,-p)\tilde{R}_0^+(p)R_0^-(p) \qquad \tilde{R}_0^+(p) = \frac{e^{-2ip}R_0^-(p)}{S_0(p,-p)\rho_1^2(p)}$

$\mathbb{D}_1$ known $\rightarrow$ $\mathbb{D}_a$ leading Lüscher $\Delta E = -\sum_{a=1}^{\infty}\int_0^{\infty}\frac{dq}{2\pi}\mathbb{D}_a\, e^{-2\tilde{\epsilon}_a L}$

$(2\dot{2})$ particle $|2\ldots2\rangle$ ground state

$\Lambda(p)_2^{su(2)} = e^{2ip}(\rho_1\Lambda_1 + \rho_2\Lambda_2 - \rho_3\Lambda_3 - \rho_4\Lambda_4) \quad \rightarrow \quad \mathbb{D}_a|_{2\dot{2}} = \left(\frac{z^{[a]}}{z^{[-a]}}\right)^4 \mathbb{D}_a|_{1\dot{1}}$

$L = 2$ smallest among non-BPS operators $\sim \mathcal{O}_Y(Z\,\Phi^{a\dot{a}}\,Z^{L-1})$

$$2\dot{2} \quad L = 2 \quad p = \frac{\pi}{2} \qquad \Delta E_{2\dot{2}}(\tfrac{\pi}{2}) = -g^{12}\cdot 2^6\left(21\zeta(9) - 3\cdot 2^2\zeta(5)\right)$$

$(1\dot{1})$ particle

$$1\dot{1} \qquad L=2 \qquad p_n=\{\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4}\}$$

$$\Delta E_{1\dot{1}}(\frac{\pi}{2}) = -2^6\cdot g^{20}\left(7\cdot 2^5\zeta(9) - 429\cdot 2^2\zeta(13) + 2431\zeta(17)\right)$$

$$\begin{aligned}\Delta E_{1\dot{1}}(\frac{\pi}{2}\mp\frac{\pi}{4}) \;&=\; -2^5\cdot g^{20}(-2^3\cdot 7\cdot(99\mp 70\sqrt{2})\zeta(9) - 2(6765\\ \mp 4785\sqrt{2})\zeta(11) \;&+\; 2002(\mp 5\sqrt{2}+7)\zeta(15) + (7293\mp 4862\sqrt{2})\zeta(17))\end{aligned}$$

no rational parts

Lüscher for $(2\dot{2})$ smaller powers of g

ABA, duality and generating functional

$\tilde{d}_{1,1}(p)$ eigenvalue for generic state m_1 y roots m_2 w roots (Galleas + experience)

$$\Lambda^{su(2)} = \left(\frac{x^+}{x^-}\right)^{m_1} \rho_1 \frac{R^{(-)+}}{R^{(+)+}} \left[\frac{R^{(+)+}}{R^{(-)+}} \frac{B_1^- R_3^-}{B_1^+ R_3^+} - \frac{B_1^- R_3^-}{B_1^+ R_3^+} \frac{Q_2^{++}}{Q_2} - \frac{u^+}{u^-} \frac{R_1^+ B_3^+}{R_1^- B_3^-} \frac{Q_2^{--}}{Q_2} + \frac{u^+}{u^-} \frac{B^{(-)-}}{B^{(+)-}} \frac{R_1^+ B_3^+}{R_1^- B_3^-} \right]$$

type 1 roots $x^+(p) = y$ type 2 $u = w_l$ type 3 $x^-(p) = y^{-1}$ Bethe eq.s:

$$\left. \frac{R^{(+)+} Q_2}{R^{(-)+} Q_2^{++}} \right|_{x^+(p)=y} = 1 \quad \left. \frac{u^-}{u^+} \frac{Q_1^- Q_2^{++}}{Q_1^+ Q_2^{--}} \right|_{u=w_l} = -1 \quad \left. \frac{B^{(-)-} Q_2}{B^{(+)-} Q_2^{--}} \right|_{x^-(p)=y^{-1}} = 1$$

interpretation: $\bullet$ massive y and $\circ$ "magnons" $S_{\bullet y}(p, y)$ etc. known

BBY for y magnons $R_y^-(y) = R_y^+(-y)$ $R_y^{\pm}(y) \equiv 1$

BBY for $\circ$ magnons $R_\circ^-(w) = R_\circ^+(-w)$ $R_\circ^{\pm}(w) \equiv 1$

from $\Lambda^{su(2)} \longrightarrow \mathcal{W}_{su(2)}^{-1}$ for generic case dualize y roots in DTM

$q(x) = x^{2m_2}\left[R^{(+)}Q_2^- - R^{(-)}Q_2^+\right]$ degree $2N + 4m_2$

m_1 roots y_j m_1 roots $-y_j$ $\tilde{m}_1$ roots $\tilde{y}_j$ $\tilde{m}_1$ roots $-\tilde{y}_j$

$\tilde{m}_1 = N + 2m_2 - m_1$ $q(x) = \gamma B_1 R_3 \tilde{B}_1 \tilde{R}_3$

$\tilde{B}_1\tilde{R}_3$ B_1R_3 with $m_1 \to \tilde{m}_1$ $y_j \to \tilde{y}_j$

$$F(x) \equiv \frac{x^{2m_2}}{B_1R_3\tilde{B}_1\tilde{R}_3}\left[R^{(+)}Q_2^- - R^{(-)}Q_2^+\right] \qquad x \text{ independent}$$

eigenvalue in the $sl(2)$ grading

$$\Lambda^{sl(2)} = \left(\frac{x^+}{x^-}\right)^{m_1-2m_2}\frac{R^{(+)-}}{R^{(+)+}}\rho_1\left[\frac{\tilde{B}_1^+\tilde{R}_3^+}{\tilde{B}_1^-\tilde{R}_3^-}\frac{Q_2^{--}}{Q_2} - \frac{R^{(-)-}}{R^{(+)-}}\frac{\tilde{B}_1^+\tilde{R}_3^+}{\tilde{B}_1^-\tilde{R}_3^-}\right.$$
$$\left. - \frac{u^+}{u^-}\frac{B^{(+)+}}{B^{(-)+}}\frac{\tilde{R}_1^-\tilde{B}_3^-}{\tilde{R}_1^+\tilde{B}_3^+} + \frac{u^+}{u^-}\frac{\tilde{R}_1^-\tilde{B}_3^-}{\tilde{R}_1^+\tilde{B}_3^+}\frac{Q_2^{++}}{Q_2}\right]$$

$\Lambda^{sl(2)} \quad \to \quad \mathcal{W}_{sl(2)}^{-1}$ $\tilde{\mathcal{W}}_{sl(2)}^{-1} = \tilde{\mathcal{W}}_{su(2)}^{-1}$